Binary Decision Diagrams

Binary Decision Diagrams (BDDs)

[Bryant 86]

- Reduced Ordered Binary Decision Diagrams (BDDs)
  - An efficient data structure for representing Boolean functions (or truth sets of Boolean formulas) and manipulating them
  - There are BDD packages available: (for example CUDD from Colorado University)

- BDDs are a canonical representation for Boolean functions
  - given two Boolean logic formulas F and G, if F and G are equivalent (i.e. if their truth sets are the same), then their BDD representations will be the same

BDDs for Symbolic Model Checking

- BDD data structure can be used to implement the symbolic model checking algorithm we discussed earlier

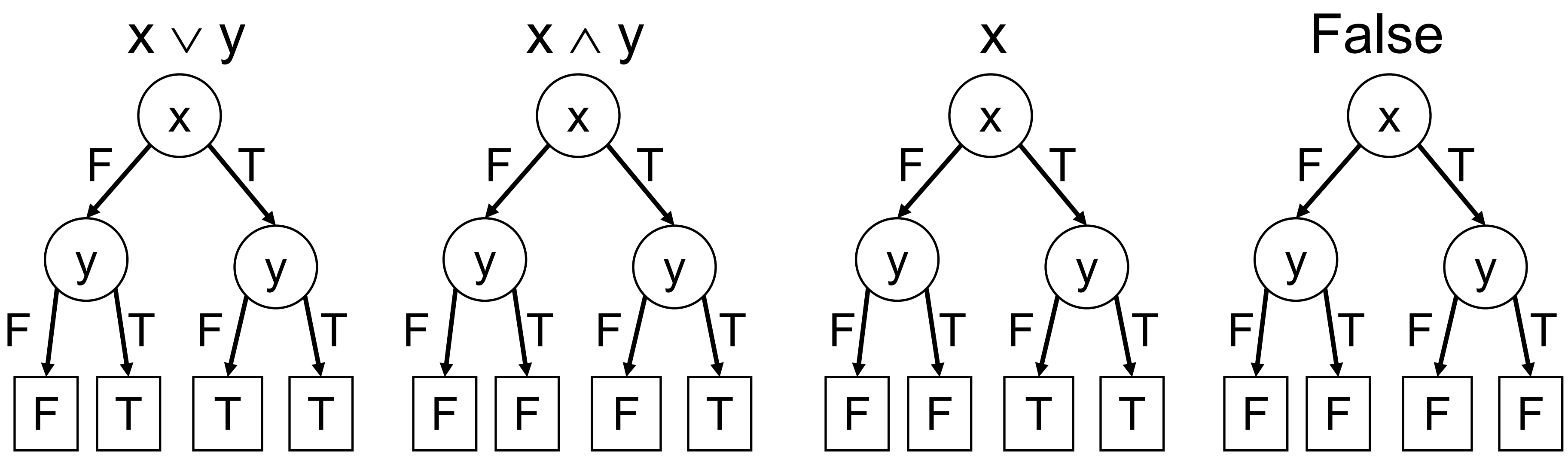
- BDDs support all the operations we need for symbolic model checking
  - take conjunction of two BDDs
  - take disjunction of two BDDs
  - test equivalence of two BDDs
  - test subsumption between two BDDs
  - negate a BDD
  - test if a BDD satisfiable
  - test if a BDD is a tautology
  - existential variable elimination

Binary Decision Trees

Given a variable order, in each level of the tree, branch on the value of the variable in that level.

- Examples for boolean formulas on two variables

  Variable order: x, y



Reduced and Ordered Binary Decision Diagrams

- We are interested in ***Reduced*** and ***Ordered*** Binary Decision Diagrams

- Reduced:
  - Merge all identical sub-trees in the binary decision tree (converts it to a directed-acyclic graph)
  - Remove redundant tests (if the false and true branches for a node go to the same place, remove that node)

- Ordered
  - We pick a fix order for the Boolean variables:

    $x_0 < x_1 < x_2 < \ldots$
  - The nodes in the BDD are listed based on this ordering

BDDs

- Repeatedly apply the following transformations to a binary decision tree:

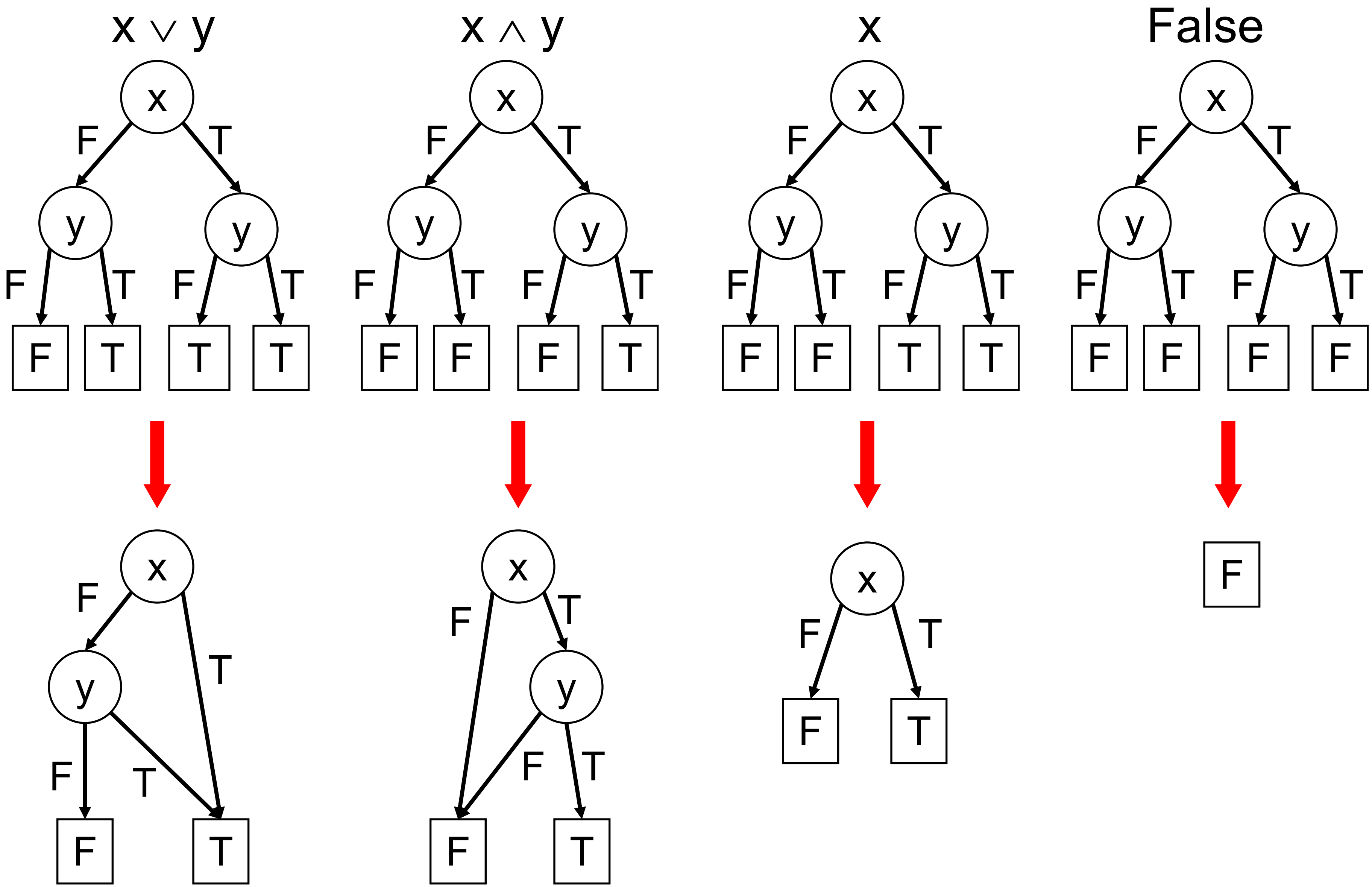
    1. Remove duplicate terminals

    2. Remove duplicate non-terminals

    3. Remove redundant tests

- These transformations transform the tree to a directed acyclic graph

Binary Decision Trees vs. BDDs



Good News About BDDs

- Given BDDs for two boolean logic formulas F and G
  - The BDDs for $F \wedge G$ and $F \vee G$ are of size $|F| \times |G|$ (and can be computed in that time)
  - The BDD for $\neg F$ is of size $|F|$ (and can be computed in that time)
  - $F \equiv ?\ G$ can be checked in linear time
  - Satisfiability of F can be checked in constant time
    - No, this does not mean that you can solve SAT in constant time

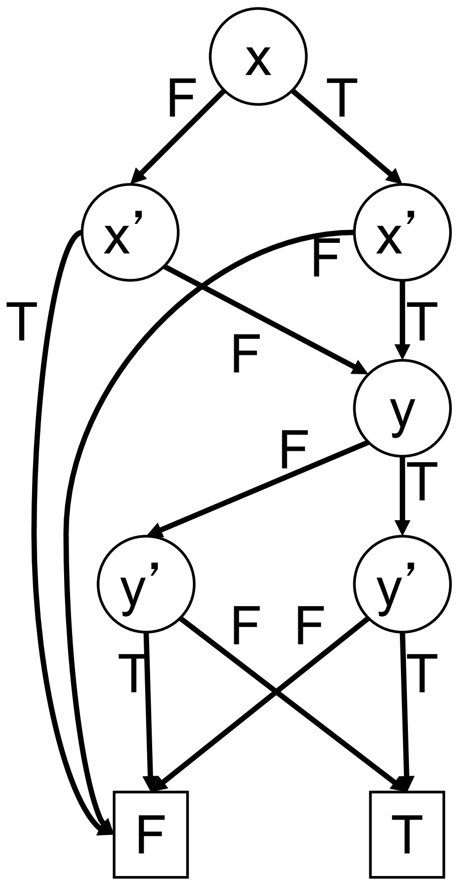
Bad News About BDDs

- The size of a BDD can be exponential in the number of boolean variables
- The sizes of the BDDs are very sensitive to the variable ordering. Bad variable ordering can cause exponential increase in the size of the BDD
- There are functions which have BDDs that are exponential for any variable ordering (for example binary multiplication)

BDDs are Sensitive to Variable Ordering

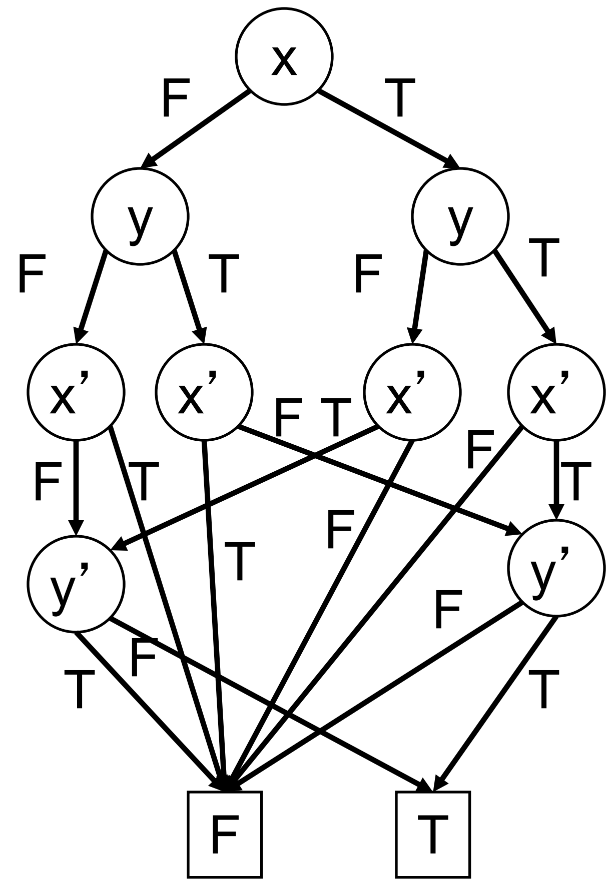
Identity relation for two variables: $(x' \leftrightarrow x) \wedge (y' \leftrightarrow y)$

Variable order: x, x’ , y, y'



For n variables, 3n+2 nodes

Variable order: x, y, x’ , y'



For n variables, $3 \times 2^n - 1$ nodes

BDDs from Another Perspective

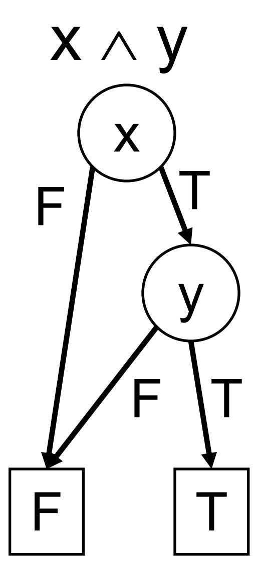
- Any Boolean formula f on variables $x_1, x_2, \ldots, x_n$ can be written as (called Shannon expansion):

$$f = x_i \wedge f\,[True/x_i] \vee \neg\, x_i \wedge f\,[False/x_i] \quad \text{(this is an if-then-else)}$$

- BDDs use this idea

This node corresponds to the formula False, which comes from the Shannon expansion:

$$False \equiv x \wedge y\,[False/x]$$



This node corresponds to the formula y, which comes from the Shannon expansion:

$$y \equiv x \wedge y\,[True/x]$$

Model counting with BDDs

- Once you construct a BDD, you can count the number of models by counting paths of the BDD
- Count the paths that reach from the root to the “True” leaf node
- You need to take into account the variables that are not represented in the BDD
  - they are removed as redundant tests but we need to keep track of them to count
- Count the number of paths that reach True
  - keep track of missing (redundant) variables on a path, and add 2^k to the count for each path that has k missing variables
- Can compute the count in linear time by traversing the nodes from leaves towards the root node