Discussion 3: Practice Proofs

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1.How many slices of pizza can one obtain by making n straight line cuts with a knife?
Or, academically, what is the max number of regions defined by n lines in the plane?

Solution:

The idea is that when we add the nth line to the plane, we can add at most n regions. Because the nth line can intersect with at most n-1 lines and the n-1 intersect points partition the line into n intervals. Each of the interval partition one region into two.

$f(n) = f(n-1) + n$

$f(1) =2$
$f(2) = f(1) + 2$
$f(3) = f(2) + 3$
...
$f(n) = f(n-1) + n$

$f(n) = (n^2+n+2)/2$

2.You are given an ordered list of n words. The length of the ith word is wi, that is the ith word takes up wi spaces. (For simplicity assume that there are no spaces between words.) Your goal is to break this ordered list of words into lines; this is called a layout. Note that you can not reorder the words.

The length of a line is the sum of the lengths of the words on that line. The ideal line length is L. No line may be longer than L, although it may be shorter. The penalty for having a line of length K is L-K.

The total penalty of a layout it the sum of the line penalties. Your problem is to design a layout that minimizes the total penalties.

Prove or disprove that the following greedy algorithm correctly solves this problem.

For i=1 to n
Place the ith word on the current line if it fits,
otherwise place the ith word on a new line.

Proof:
This algorithm is correct for the problem of minimizing the total sum of all line penalties. The proof is by contradiction. Assume there is an optimal solution T, and call the output of the greedy algorithm G. Let si be the penalty of the ith line of solution S. Let j be the number of the first line in T that is different from jth line in G. By the definition of the algorithm, gi < ti. Create a new solution T' by moving the first word of line i+1 in T to the end of line i. Let l be the length of this word. Note that t'_{i+1} = t_{i+1} + l and t'_i = t'_i – l.
Therefore, the sum of all line penalties in T' is the same as the sum of all line penalties of T. But T' is more like greedy than T, and has the same total penalties. Contradiction.