Chapter 11: Decidability *

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- The corresponding textbook chapter should be read before attending this lecture.
- These notes are not intended to be complete. They are supplemented with figures, and other material that arises during the lecture period in response to questions.

*Based on **Theory of Computing**, 2nd Ed., D. Cohen, John Wiley & Sons, Inc.

Definitions

- A problem is **effectively solvable** if there is an *algorithm* for solving it (a procedure that completes after finitely many steps, the maximum of which is known in advance, but may depend on the size of the input).
- A problem whose solution is "yes" or "no" is a **decision problem**.
- An effective solution to a decision problem is a **decision procedure**.
- A decision problem that has a decision procedure is **decidable**.

Theorem: Let A be an FA. The question "Is $L(A) = \emptyset$?" is decidable.

Proof:

1. $L(A) \neq \emptyset$ if and only if there is a path from A's start state to some final state.
2. The following algorithm returns true if and only if there is a path from A's start state to some final state.

$boolean\ isEmpty(FA\ A)$ {

paint q_0 blue;
$set.put(q_0)$;
while (! $set.isEmpty()$) {
  $p = set.remove()$;
  For each $a \in \Sigma$,
    if ($p' = \delta(p, a)$ is not blue) {
      paint p' blue;
      $set.put(p')$;
    }
}
return (there is a blue final state) ? false : true;

}

Theorem: Let A be an FA with n states. If $L(A) \neq \emptyset$, A accepts a word w, $|w| < n$.

Proof:

1. The shortest path from A's start state to some final state, if any, can be no longer than $n - 1$: It cannot involve a circuit.
2. The concatenation of arc labels consists of less than n letters.

Thus, "Is $L(A) = \emptyset$?" also can be answered by running A on no more than

$$m^{n-1} + m^{n-2} + \cdots + m^0$$

words, where $m = |\Sigma|$.

THEOREM: Let A and B be FA accepting L_A and L_B, respectively, and E and F be regular expressions. The following questions are decidable:

1. Is $L_A = \emptyset$?
2. Is $L_A = L_B$?
3. Is E equivalent to F (i.e., do they denote the same language)?

PROOF:

1. This follows from our previous theorem.
2. $L_A = L_B \iff (L_A \cap \overline{L_B}) \cup (L_B \cap \overline{L_A}) = \emptyset$.
3. For each regular expression, construct an equivalent FA , using Kleene's theorem. Use part 2 above to see if these 2 FA accept the same language.

Finiteness

Theorem: Let R be a regular expression. $|L(R)| = \infty \iff R$ has a Kleene star that applies to something other than Λ.

Proof:

1. If R has no Kleene star operator, it denotes a finite set.
2. $\Lambda^* = \Lambda$.
3. If the Kleene star operator is applied to something not equivalent to Λ, the resulting set is infinite.

Theorem: Let A be an FA with n states.

$$L(A) = \infty \iff \exists w \in L(A), n \le |w| < 2n.$$

Proof:

1. If $\exists w \in L(A), n \le |w| < 2n$, then $L(A) = \infty$.
   This follows from the pumping lemma: If there is any word w, $|w| \ge n$, then $w = xyz$, such that $xy^k z \in L(A)$, $\forall k > 0$.
2. If $L(A) = \infty$ then $\exists w \in L(A), n \le |w| < 2n$.
   (a) By the pumping lemma, $\exists w \in L(A), w = xyz$, and $|xy| \le n$.
   (b) Thus, $|y| \le n$.
   (c) Assume without loss of generality that the part of the accepting path associated with x and z do not contain any loops.
   (d) $xz \in L(A)$, and $|xz| < n$.
   (e) Let k be the *smallest* exponent that makes $|xy^k z| \ge n$. Then, $|xy^k z| < 2n$.

Theorem: Given FA A, the question "Is $L(A) = \infty$?" is decidable.

Proof:

1. Check each word w, $n \leq |w| < 2n$.
2. If any are accepted, $L(A) = \infty$;
   otherwise, $L(A) < \infty$.

When you study DFS, you should conceive of faster ways to test for finiteness.

However, the proof above, requires only that:

- you know how many states the FA has;
- you know if $w \in L(A)$, for any word w.

You do not need to be able to *examine* the FA.