Chapter 4: Properties of Regular Languages*

Peter Cappello
Department of Computer Science
University of California, Santa Barbara
Santa Barbara, CA 93106
cappello@cs.ucsb.edu

• Please read the corresponding chapter before attending this lecture.

• These notes are supplemented with figures, and material that arises during the lecture in response to questions.

• Please report any errors in these notes to cappello@cs.ucsb.edu. I’ll fix them immediately.

*Based on An Introduction to Formal Languages and Automata, 3rd Ed., Peter Linz, Jones and Bartlett Publishers, Inc.
4.1 Closure Properties of Regular Languages

Closure under Simple Set Operators

**Thm. 4.1:** If $L_1$ and $L_2$ are regular languages, then so are $L_1 \cup L_2$, $L_1 \cap L_2$, $L_1 L_2$, $\overline{L_1}$, and $L_1^*$.

**Proof:**

1. Assume that $L_1$ and $L_2$ are regular.

2. Let regular expression $r_1$ and $r_2$ denote $L_1$ and $L_2$, respectively.

3. Then,
   - $r_1 + r_2$ denotes $L_1 \cup L_2$,
   - $r_1 r_2$ denotes $L_1 L_2$,
   - $r_1^*$ denotes $L_1^*$.

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4. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA that accepts $L_1$.

5. Then, $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$ accepts $\overline{L_1}$.

6. Since regular languages are closed under complement and union, $\overline{L_1} \cup \overline{L_2} = L_1 \cap L_2$ is a regular language.
• Let $w = s_1 s_2 \cdots s_n$ be a word over $\Sigma$. Then, $w^R$ denotes the word $s_n \cdots s_2 s_1$, the reverse of $w$. $\lambda^R = \lambda$.

• Let $L$ be a language. Then $L^R$ denotes $L^R = \{w^R : w \in L\}$, called the reverse of $L$.

**Thm. 4.2:** The family of regular languages is closed under reversal.

**Proof:**

1. Let $L$ be regular, and $M = (Q, \Sigma, \delta, q_0, \{q_f\})$ be an NFA that accepts it\(^1\).

2. We construct $M^R = (Q, \Sigma, \delta^R, q_f, \{q_0\})$, where $\delta^R$ is $\delta$ with the orientation of the arcs reversed.

3. There is a path from $q_0$ to $q_f$ in $M$ if and only if there is a path from $q_f$ to $q_0$ in $M^R$: $L(M^R) = L^R$.

\(^1\)We may assume without loss of generality that $|F| = 1$
Closure under Other Operators

**Def. 4.1:** Let $\Sigma$ and $\Gamma$ be alphabets. Then, a function

$$h : \Sigma \mapsto \Gamma^*$$

is called a **homomorphism**.

- For each symbol in $\Sigma$, a homomorphism substitutes a word in $\Gamma^*$.
- Let $w = s_1s_2 \cdots s_n$. Then,

  $$h(s_1s_2 \cdots s_n) = h(s_1)h(s_2) \cdots h(s_n).$$

- If $L$ is a language on $\Sigma$, then its homomorphic image is

  $$h(L) = \{h(w) : w \in L\}.$$
Example:

- Let $\Sigma = \{0, 1\}$ and $\Gamma = \{a, b, \ldots, z\}$.
- Define $h$ as follows:

  $$
  h(0) = \text{hello} \\
  h(1) = \text{goodbye}
  $$

- Then, $h(010) = \text{hellogoodbyehello}$.
- The homomorphic image of $L = \{00, 010\}$ is

  $$
  h(L) = \{\text{hellohello, hellogoodbyehello}\}.
  $$
Thm. 4.3: Let $h$ be a homomorphism. If $L$ is a regular language, then its homomorphic image $h(L)$ is regular. The family of regular languages therefore is closed under arbitrary homomorphisms.

Proof:

1. Assume that $L$ is regular, and let $M$ be a DFA that accepts $L$.

2. Construct a generalized transition graph (GTG), based on the transition graph (TG) for $M$ as follows:
   For each symbol, $s$, that labels an arc in the TG for $M$, label that same arc in the GTG with $h(s)$.

3. There is a path labelled $w$ from $q_0$ to some final state $q_f$ in the TG for $M$ if and only if there is a path labelled $h(w)$ from $q_0$ to $q_f$ in the GTG.
Def. 4.2: Let $L_1$ and $L_2$ be languages on the same alphabet. Then, the right quotient of $L_1$ with $L_2$ is defined as

$$L_1/L_2 = \{x : xy \in L_1 \text{ for some } y \in L_2\}.$$ 

Example: If

$$L_1 = \{a^nb^m : n \geq 1, m \geq 0\} \cup \{ba\}$$

and

$$L_2 = \{b^m : m \geq 1\},$$

then

$$L_1/L_2 = \{a^nb^m : n \geq 1, m \geq 0\}.$$ 

• Draw a TG for $L_1$.

• Identify each state, $q_i$ in $TG_1$ such that there exists a $y \in L_2$ and there is a path from $q_i$ to a final state in $TG_1$. 

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• There are 2 such states, $q_1$ and $q_2$.
  These are the final states in $L_1/L_2$. 
**Thm. 4.4:** If $L_1$ and $L_2$ are regular languages, then $L_1/L_2$ is regular: The family of regular languages is closed under right quotient with a regular language.

**Proof:**

1. Assume that $L_1$ and $L_2$ are regular, and let DFA $M = (Q, \Sigma, \delta, q_0, F)$ accept $L_1$.

2. We construct DFA $\hat{M} = (Q, \Sigma, q_0, \hat{F})$ as follows.
   
   (a) For each $q_i \in Q$, determine if there is a $y \in L_2$ such that $\delta^*(q_i, y) \in F$.

   (b) This can be done by the following procedure:
      
      i. Construct $M_i = (Q, \Sigma, \delta, q_i, F)$.
         
      ii. If $L_2 \cap L(M) \neq \emptyset$ then $q_i \in \hat{F}$.

3. If $x \in L_1/L_2$ then $x \in L(\hat{M})$. 

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4. If $x \in L_1/L_2$, there exists a $y \in L_2$ such that $xy \in L_1$.

5. If $xy \in L_1$, then:
   - $\delta(q_0, x) = q$, for some $q \in Q$
   - $\delta(q, y) \in F$
   - By construction, $q \in \widehat{F}$, so $\widehat{M}$ accepts $x$.

6. It similarly is easy to show that If $x \in L(\widehat{M})$ then $x \in L_1/L_2$. 