CS 138: Formal languages and Automata

Homework 1 solutions

1. • $S_1 \times S_2 = \{(2, 2), (2, 4), (2, 5), (2, 8), (2, 9), (3, 2), (3, 4), (3, 5), (3, 8), (3, 9), (5, 2), (5, 4), (5, 5), (5, 8), (5, 9), (7, 2), (7, 4), (7, 5), (7, 8), (7, 9)\}$
   • $S_2 \times S_1 = \{(2, 2), (2, 3), (2, 5), (2, 7), (4, 2), (4, 3), (4, 5), (4, 7), (5, 2), (5, 3), (5, 5), (5, 7), (8, 2), (8, 3), (8, 5), (8, 7), (9, 2), (9, 3), (9, 5), (9, 7)\}$

2. $|S \cap T| + |S \cup T| = \{2, 6, 8\} + \{2, 4, 5, 6, 8\} = 3 + 5 = 8$

3. $S_1 \subseteq S_2$ by definition $\Rightarrow x \in S_1 \rightarrow x \in S_2$ contrapositive $x \notin S_2 \rightarrow x \notin S_1 \Rightarrow x \in \bar{S}_2 \rightarrow x \in \bar{S}_1$

4. $(S_1 \cap \bar{S}_2) \cup (\bar{S}_1 \cap S_2) = \emptyset$ and $\bar{S}_1 \cap S_2 = \emptyset$ and $S_1 - S_2 = \emptyset$ and $S_2 - S_1 = \emptyset$

5. Induction of $|v|$
   - **Basis step:** $|v| = 0$, $(u\lambda)^R = u^R$ and $\lambda R u^R = u^R$.
   - **Inductive hypothesis:** $(uv)^R = v^R u^R$ holds for all $v$ of length 1, 2, ..., $n$.
   - **Inductive step:** We want to show that $(uv)^R = v^R u^R$ where $|v| = n + 1$. Let $v = \omega a$ where $|\omega| = n$, so $(uv)^R = (u\omega a)^R = ((u\omega)a)^R = (\text{using the question hint}) a^R (u\omega) a^R = (\text{using induction hypothesis}) a^R v^R a^R = v^R u^R$

6. • abaabaabaa, aaaaabaaa, baaaaaab
   • aaaaaabaa, baaaaaba

7. $G = \{(S), \{a\}, S, P\}$ and $P$ is $S \rightarrow aaS, S \rightarrow \lambda$.

8. • 1st: $\lambda$, 2nd: $a$, 3rd: $aa, ..., 129$st: $a^{128}$
   - If $|\omega| = k$, we have $k^n$ words with length $\omega$, e.g., if $\omega = 0$, there is $2^0 = 1$ word $\omega$ and if $\omega = 1$, there are totally $2^1 = 2$ words ($a$ and $b$). On the other hand $513 = 2^6 + 2^4 + 2^2 + 2$, so 127st word is $b^6$, 128st word is $a^7$, and 129st word is $a^6 b$.

9. **Basis step:** if $|u| = 1$ then $u$ is a single symbol in $\Sigma$ and it’s obvious that $|u^2| = 2|u|$
   - **Inductive hypothesis:** $|u^2| = 2|u|$ holds for all $u$ of length 1, 2, ..., $n$.
   - **Inductive step:** We want to show that $|u^2| = 2|u|$ where $|u| = n + 1$. Let write $u$ as $\omega a$ where $|\omega| = n$ and $a$ is a single symbol, then $|u^2| = |\omega\omega a| = |\omega^2| + |aa| = 2n + 2 = 2|u|$
10. We first show that all sentential forms must have the form \( \omega_i = a^{2i} Sb^i \). Suppose that it holds for all sentential forms \( \omega_i \) of length \( 3i + 1 \) or less. The only way to get another sentential form is to apply the production \( S \rightarrow aaSb \). It gets us \( a^{2i} Sb^i \Rightarrow a^{2i+2} Sb^{i+1} \), so that every sentential form of length \( 3i + 4 \) is also of form \( \omega_i = a^{2i} Sb^i \).

Finally, to get a sentence, we must apply the production \( S \rightarrow \lambda \), so \( S \Rightarrow a^{2n} Sb^n \Rightarrow a^{2n}b^n \) represents all possible derivations. Thus, \( G \) can derive only strings of the form \( a^{2n}b^n \). To show that all strings of this form can be derived we simply apply \( S \rightarrow aaSb \) as many times as needed, followed by \( S \rightarrow \lambda \).