Homework 2, due Wednesday January 23

1. (1 point each part) For each of the following constant-coefficient systems \( y' = Ay \), determine if the system is stable, asymptotically stable, or unstable:

(a) \[
A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

(b) \[
A = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix}
\]

(c) \[
A = \begin{pmatrix} -1 & 100 \\ 0 & -2 \end{pmatrix}
\]

(d) \[
A = \begin{pmatrix} 0 & 10 \\ -5 & 0 \end{pmatrix}
\]

2. (5 points) Consider the method of lines applied to the diffusion equation in one space dimension, \( u_t = au_{xx} \), with \( a > 0 \) a constant, \( u = 0 \) at \( x = 0, x = 1 \) for \( t \geq 0 \), and with given initial values. Formulate the method of lines using the central difference approximation to the derivative \( u_{xx} \), as in Example 1.3 of the text (as we did in Lecture 1), to arrive at a linear constant coefficient ODE system \( y' = Ay \) with \( A \) symmetric. Find the eigenvalues of \( A \) to determine whether the problem is unstable, stable or asymptotically stable. (Hint: Try eigenvector \( v^k_i \) in the form \( v^k_i = \sin(ik\pi\Delta x) \), \( i = 1, \ldots, m \), for \( 1 \leq k \leq m \), and use the sum of angles formula for \( \sin \)).

3. (5 points) Consider the method of lines applied to the advection equation in one space dimension, \( u_t + au_x = 0 \), with \( a > 0 \) a constant, on the spatial domain \( 0 \leq x \leq 1 \), with boundary condition \( u = 0 \) at
$x = 0$, and with given initial values. Formulate the method of lines using the first order backward difference approximation to the derivative $u_x$, to arrive at a linear constant coefficient ODE system $y' = Ay$. Find the eigenvalues of $A$ to determine whether the problem is unstable, stable or asymptotically stable. Next, consider approximating the derivative $u_x$ by the first order forward difference approximation (see the Scholarpedia MOL article for the definition of the forward difference approximation). For this, use the boundary condition $u = 0$ at $x = 1$. Find the eigenvalues of $A$ to determine whether the problem is unstable, stable or asymptotically stable.

4. (6 points) Write a well-structured MATLAB code implementing the explicit Euler method for systems of standard-form ODEs. Apply the code to the following problem. (Note that the system is uncoupled, but you are to treat it in your code as if it were coupled.)

$$
\begin{align*}
y'_1 &= -y_1 \\
y'_2 &= -100(y_2 - \sin(t)) + \cos(t)
\end{align*}
$$

for $0 \leq t \leq 1$, with initial value $y_1 = 1$, $y_2 = 2$. Try this for 2 sequences of stepsizes:

- $h = .01$
- $h = .05$

Plot the solutions obtained. Explain your results.