Efficient Power Control via Pricing in Wireless Data Networks*

Cem U. Saraydar Narayan B. Mandayam†
WINLAB
Rutgers University
73 Brett Road
Piscataway, NJ 08854-8060
saraydar,narayan@winlab.rutgers.edu

David J. Goodman
Dept. of Electrical Engineering
Polytechnic University
6 Metrotech Center
Brooklyn, NY 11201, USA
dgoodman@duke.poly.edu

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Abstract

A major challenge in operation of wireless communications systems is the efficient use of radio resources. One important component of radio resource management is power control, which has been studied extensively in the context of voice communications. With increasing demand for wireless data services, it is necessary to establish power control algorithms for information sources other than voice. We present a power control solution for wireless data in the analytical setting of a game theoretic framework. In this context, the quality of service (QoS) a wireless terminal receives is referred to as the *utility* and distributed power control is a *non-cooperative power control game* where users maximize their utility. The outcome of the game results in a *Nash equilibrium* that is inefficient. We introduce pricing of transmit powers in order to obtain Pareto improvement of the non-cooperative power control game, i.e. to obtain improvements in user utilities relative to the case with no pricing. Specifically, we consider a pricing function that is a linear function of the transmit power. The simplicity of the pricing function allows a distributed implementation where the price can be broadcast by the base station to all the terminals. We see that pricing is especially helpful in a heavily loaded system.

1 Introduction

As the demand for wireless services increases, efficient use of resources grows in importance. A fundamental component of radio resource management is transmitter power control. It is well known that minimizing interference using power control increases capacity [1–3] and also extends battery life. Recently, an alternative approach to the power control problem in wireless systems based on an economic model has been offered [4–7]. In this model, service preferences for each user are represented by a utility function. As

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the name implies, the utility function quantifies the level of satisfaction a user gets from using the system resources. Game-theoretic methods are applied to study power control under this new model. Game theory is a powerful tool in modeling interactions between self-interested users and predicting their choice of strategies [8–10]. Each player in the game maximizes some function of utility in a distributed fashion. The game settles at a Nash equilibrium if one exists. Since users act selfishly, the equilibrium point is not necessarily the best operating point from a social point of view. Pricing the system resources appears to be a powerful tool for achieving a more socially desirable result.

Although work on pricing in computer networks can be found in the literature [11–13], pricing in wireless networks remains a topic to be explored. MacKie-Mason and Varian [11] study the effects of pricing on the way it determines users' demand of the resources. Capacity under the optimal pricing scheme is also studied. Cocchi, et. al [12] study pricing policies in multiple service class computer networks. Specifically, the interaction between pricing and multiple qualities of service and the user incentives offered by the network as a result of this interaction are the foci of attention.

In this work, we are primarily concerned with the impact of pricing the usage of wireless services on QoS. Pricing of services in wireless networks emerges as an effective tool for radio resource management because of its ability to guide user behavior towards a more efficient operating point. To that end, we introduce a model for power control in wireless data networks using concepts from microeconomics. We model utility to reflect the level of satisfaction (QoS) a data user gets from using system resources [4]. We consider the uplink power control problem in a single cell CDMA wireless data system with n users where each user maximizes its own utility. While the resulting noncooperative power control game has a Nash equilibrium, it is inefficient. Therefore, we introduce pricing to improve efficiency. We then show that there exist equilibria in the non-cooperative power control game with pricing and that they are Pareto superior compared to the equilibrium of the game with no pricing. However, the game with pricing is still unable to achieve a socially optimum power solution.

This paper is organized as follows. In Section 2, we discuss the concept of utility and develop a utility function that represents the QoS of data users. In Section 3, we construct the non-cooperative power control game. The equilibrium properties are discussed in Sections 4 and 5. Section 6 is devoted to showing the inefficiency of the Nash equilibrium obtained as a result of the non-cooperative power control game. Pareto improvement is achieved by the game with pricing which is discussed in Section 7. In Section 7.1, we define supermodular games and discuss the relevance of this class of games in the context of the present work. Comparisons of results in games with and without pricing and the significance of these results are discussed in Section 8. In Section 9, we define the social optimum and discuss how it relates to solutions from the games discussed in this work. Finally, in Section 10 we present an overview of the results in this paper and conclusions.

2 Utility Function

The concept of utility is commonly used in microeconomics and refers to the level of satisfaction the decision-taker receives as a result of its actions. A utility function describes the preference relation between the elements of an individual's set of choices. A more desirable or preferred course of action returns a higher utility value. Formally, a utility function is defined as follows.

Definition 1 *A function that assigns a numerical value to the elements of the action set A ($u : A \to \Re^1$) is a utility function, if for all $x, y \in A$, x is at least as preferred compared to y if and only if $u(x) \geq u(y)$.*

The utility function that describes a particular set of preference rules is not unique. Any function that puts the elements of A in the desired order is a candidate for a utility function. We first identify the preference relations that are specific to our problem and then construct a utility function that satisfies this structure.

Users access a wireless system through the air interface which is a common resource and they transmit information expending battery energy. Since the air interface is a shared medium, each user's transmission is a source of interference for others. The signal-to-interference ratio (SIR) is a measure of the quality of signal reception for the wireless user. Typically, a user would like to achieve a high quality of reception (high signal-to-interference ratio, SIR) while at the same time expending a small amount of energy. Thus, it is possible to view both SIR and battery energy (or equivalently transmit power) as commodities that a wireless user desires. A user yearns for a high SIR while trying to conserve energy. The SIR a user achieves at its base station increases with an increase in its own power and decreases with an increase in power of any other user. Thus, there exists a trade-off relationship between obtaining high SIR and low energy consumption. Finding a good balance between the two conflicting objectives is the primary focus of the power control component of radio resource management.

An optimum power control algorithm for wireless voice systems maximizes the number of conversations that can simultaneously achieve a certain quality of service (QoS) objective; typically a minimum acceptable SIR. However, this approach is not appropriate for the efficient operation of a wireless data system [4, 14]. This is because the QoS objective for data signals differs from the QoS objective for telephones. Figure 1 shows the utility function for voice systems. A voice connection is considered unacceptable if the SIR is below a certain threshold. It is also assumed that a voice user is indifferent to the changes in the SIR so long as it remains above the threshold. Hence, the step-function characteristic of utility for voice systems.

In a data system, error-free communication has high priority. The SIR is an important quantity since there is a direct relationship between the SIR and the probability of transmission errors. When a data system detects an error in transmission, the data in error has to be retransmitted. Consider a cellular system where each user transmits L information bits in frames of $M > L$ bits at a rate R bits/second using p Watts of power. Let P_c denote the probability of correct reception of a frame at the receiver, i.e. the frame success rate (FSR). P_c is a function of the SIR obtained by the terminal at its base station and depends on the properties of the system such as modulation, radio propagation and receiver structure. If the number of transmissions necessary to receive a packet correctly is denoted by K, then the probability mass function of the random variable K can be expressed as

$$P_K(k) = \begin{cases} P_c(1-P_c)^{k-1} & \text{for } k = 1, 2, \cdots, K \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where the transmissions are assumed to be statistically independent. The expected value of K is $E[K] = \frac{1}{P_c}$. The total transmission time required for successful reception is $\frac{KM}{R}$ seconds and the total energy expended is $\frac{pKM}{R}$ Joules with expected value $\frac{pM}{RP_c}$. Therefore, in order to transmit an L bit packet successfully, a terminal expends $\frac{pM}{RP_c}$ Joules of energy. Finally, the utility function can be expressed as [14]

$$u = \frac{L}{pM/RP_c} = \frac{LRP_c}{Mp} \quad \text{bits/Joule} \quad . \quad (2)$$

Utility as defined above is the number of information bits received successfully per Joule of energy expended. While the above formulation is extremely general, we examine some examples for illustrative purposes.

Recall that the frame success rate (FSR), P_c, depends on the SIR and is determined by the system characteristics. Assuming perfect error detection and no error correction, we can express the FSR as $P_c = (1 - P_e)^M$ where P_e is the bit error rate (BER). In the case of an additive white Gaussian noise (AWGN) channel, the BER expressions for various modulation techniques are given in Table 1. In all cases the BER decreases monotonically with SIR, where SIR is denoted by γ. Consequently, P_c is a monotonically increasing function of the SIR. Therefore, P_c can be expressed as a function of γ and substituted in equation (2) to obtain the utility function for a specific system. However, the utility function given in (2) has a mathematical anomaly in its formulation. In case of transmit power $p = 0$, for all modulation schemes, the best strategy for the receiver is to make a guess for each bit, resulting in $P_c = 2^{-M}$, resulting in infinite utility [6, 15]. This suggests that in order to maximize utility, all users in the system should transmit zero power and just wait for the receiver to guess the correct data. To avoid this degenerate solution, we approximate the FSR, P_c, by an *efficiency function* that closely follows the behavior of the probability of correct reception while producing $P_c = 0$ at $p = 0$. We introduce the efficiency function defined as

$$f(\gamma) = (1 - 2P_e)^M \quad . \quad (3)$$

to replace P_c in (2). The resulting utility function will be examined in the remainder of the work. It is given as

$$u = \frac{LRf(\gamma)}{Mp} \quad \text{bits/Joule} \quad . \quad (4)$$

The efficiency function yields the desirable properties $f(0) = 0$ for $p = 0$ and $f(\infty) = 1$. At any other value of the SIR, its shape follows that of P_c. Figure 2 demonstrates how closely the efficiency function follows the FSR in case of BPSK and non-coherent FSK modulation schemes. The proximity in behavior of the

BPSK	$Q(\sqrt{2\gamma})$
DPSK	$\frac{1}{2}e^{-\gamma}$
Coherent FSK	$Q(\sqrt{\gamma})$
Non-coherent FSK	$\frac{1}{2}e^{-\gamma/2}$

Table 1: The bit-error-rate (BER) as a function of signal-to-interference-ratio (SIR) for various modulation schemes

two functions leads us to expect that the transmit power that maximizes utility defined in equation (2) is close to the power that maximizes (4). In the remainder of the paper, we consider power control schemes where each data user tries to maximize its individual utility.

3 Non-cooperative Power Control Game

Let $\Gamma = [N, \{P_i\}, \{u_i(\cdot)\}]$ denote the noncooperative power control game (NPG) where $N = \{1, 2, \cdots, n\}$ is the index set for the mobile users currently in the cell, P_i is the strategy set and $u_i(\cdot)$ is the payoff function of user i. Each user selects a power level p_i such that $p_i \in P_i$. Let the power vector $\mathbf{p} = [p_1, \cdots, p_n]$ denote the outcome of the game in terms of the selected power levels of all the users. The resulting utility level for the ith user is $u_i(\mathbf{p})$. We will occasionally use an alternative notation $u_i(p_i, \mathbf{p}_{-i})$ where $\mathbf{p}_{-i}$ denotes the vector consisting of elements of $\mathbf{p}$ other than the ith element. The latter notation emphasizes that the ith user has control over its own power, p_i only. Notice that $\mathbf{p} \in P$ where $P = P_1 \times P_2 \times \cdots \times P_n$ is the strategy space formed by taking the Cartesian product of individual strategy sets. For example, if $P_i \subset \Re$ for all users then $P \subset \Re^n$.

The utility user i obtains by expending p_i can be expressed more formally as

$$u_i(p_i, \mathbf{p}_{-i}) = \frac{LR}{Mp_i} f(\gamma_i) \quad \text{bits/Joule} \quad , \tag{5}$$

where γ_i is the SIR of user i defined as

$$\gamma_i = \frac{W}{R} \frac{h_i p_i}{\sum_{j \neq i} h_j p_j + \sigma^2} \quad , \tag{6}$$

where W is the available spread spectrum bandwidth [Hz], σ^2 is the AWGN power at the receiver [Watt], and $\{h_i\}$ is the set of path gains from the mobile to the base station. We assume that the strategy space P_i of each user is a compact, convex set with minimum and maximum power constraints denoted by $\underline{p}_i$ and $\overline{p}_i$, respectively. For NPG, we let $\underline{p}_i = 0$ for all i which results in the strategy space $P_i = [0, \overline{p}_i]$.

Note that equation (5) demonstrates the strategic interdependence between users. The level of utility each user gets depends not only on its own power level but also on the choice of other players' strategies through the SIR of that user. Note that the efficiency function can be chosen to represent any given modulation technique consistent with the approximation rule described in the previous section. For our numerical examples, we use the efficiency function,

$$f(\gamma_i) = (1 - e^{-0.5\gamma_i})^M \quad , \tag{7}$$

which approximates P_c for non-coherent FSK. A comparison of the difference between P_c and $f(\gamma)$ as a function of the SIR for $M = 80$ can be found in Figure 2.

In the power control game, each user maximizes its own utility in a distributed fashion. Formally, the non-cooperative power control game (NPG) is expressed as

$$\begin{aligned} \textbf{(NPG)} \quad \max_{p_i} \quad & u_i(p_i, \mathbf{p}_{-i}), \quad \text{for all } i = 1, 2, \cdots, n \\ \text{s.t.} \quad & p_i \in P_i, \quad \text{for all } i = 1, 2, \cdots, n \end{aligned} \tag{8}$$

where u_i is given in (5) and $P_i = [0, \overline{p}_i]$ is the strategy space of user i. Since utility is a function of power level of all users in the system, there exists a strong interdependence between the course of action each user takes in order to maximize its utility. However, it is possible to characterize a set of powers where the users are satisfied with the utility they receive given the power selections of other users. We need to explore the existence of such a set of powers that will constitute an *equilibrium* of power choices for the users.

4 Nash equilibrium in NPG

The solution that is most widely used for game theoretic problems is the *Nash equilibrium* [16]. In the rest of our study, we will focus on finding or characterizing such point(s) in the strategy space.

Definition 2 *A power vector* $\mathbf{p} = [p_1, \cdots, p_n]$ *is a* Nash equilibrium *of the NPG* $\Gamma = [N, \{P_i\}, \{u_i(\cdot)\}]$ *if for every* $i \in N$,

$$u_i(p_i, \mathbf{p}_{-i}) \geq u_i(p_i', \mathbf{p}_{-i}) \tag{9}$$

for all $p_i' \in P_i$.

At a Nash equilibrium, given the power levels of other players, no user can improve its utility level by making individual changes in its power. The power level chosen by a user constitutes a *best response* to the powers actually chosen by other players. The solution to the NPG in (8) is a Nash equilibrium point, $\mathbf{p}$, if one exists. Indeed, such an equilibrium exists in the case of the NPG, $\Gamma = [N, \{P_i\}, \{u_i(\cdot)\}]$.

Theorem 1 *A Nash equilibrium exists in game* $\Gamma = [N, \{P_i\}, \{u_i(\cdot)\}]$ *if for all* $i = 1, \cdots, n$,

1. P_i *is a nonempty, convex, and compact subset of some Euclidean space* $\Re^n$.

2. $u_i(\mathbf{p})$ *is continuous in* $\mathbf{p}$ *and quasiconcave in* p_i.

The proof of the above theorem can be found in Appendix A. The analysis used in the proof uses the efficiency function that approximates the probability of correct reception of non-coherent FSK as an example. However, the result that u_i is quasiconcave in p_i applies to a fairly general class of modems. It is shown in [15] that the efficiency functions that correspond to the modulation schemes listed in Table 1 are all quasiconcave in each user's own power. The above theorem establishes the existence of a Nash equilibrium in NPG.

For a given interference vector, $\mathbf{p}_{-i}$, the power p_i that maximizes u_i is found to satisfy,

$$f'(\gamma_i)\gamma_i - f(\gamma_i) = 0, \quad i = 1, \cdots, n \quad . \tag{10}$$

Assuming that $p_i \leq \overline{p}_i$ for all i, at the point (10) is satisfied, we can derive the equilibrium powers in closed form. Note that equation (10) is obtained by the first order necessary optimality condition for the utility function. It will be clear in the discussion in the following section that in addition to the expression in (10) being the same for all users, there exists only one value of SIR that satisfies it, i.e. $\gamma_i = \tilde{\gamma}$ for all i where $\tilde{\gamma}$ is the specific SIR value that depends on the form of the efficiency function $f(\cdot)$. When all users achieve $\gamma_i = \tilde{\gamma}$ under the condition that $p_i \leq \overline{p}_i$, then the equilibrium has the interesting property stated below.

Theorem 2 *There exists a unique equal-SIR Nash equilibrium in NPG if the equilibrium powers are less than maximum power.*

The proof of the above theorem can be found in Appendix B. We already mentioned that the equilibrium SIR, $\tilde{\gamma}$, of NPG is derived from the efficiency function given in equation (3). If all the wireless terminals use the same modulation technique and the same packet length, M, they have the same efficiency function. Therefore, the value of $\tilde{\gamma}$ that each terminal tries to achieve at equilibrium is the same for all terminals. It is worth noting that the power control solution obtained at the equal-SIR NPG equilibrium is similar to the solutions offered by power control algorithms for speech communications [1, 2, 17]. In fixed-target type power control algorithms for voice systems, users adjust powers in order to satisfy a minimum target SIR constraint. The algorithm terminates at a set of powers where each terminal has exactly the target SIR. The Nash equilibrium SIR of $\tilde{\gamma}$ can be thought of as the target SIR in voice systems with one important distinction: the common target SIR for voice systems is determined by subjective measures of speech quality. However $\tilde{\gamma}$ is derived from the particular efficiency function and therefore is dictated by system properties such as modulation technique, channel model and packet length. Nevertheless, the equal-SIR equilibrium of NPG can be implemented using the SIR balancing algorithm described in [2]. This algorithm would direct each terminal to determine interference periodically and adjust its power to achieve the equilibrium SIR of $\tilde{\gamma}$. After each adjustment, the other terminals adjust their powers in the same way. As a result of repeating this procedure as described, all powers converge to values that correspond to the equilibrium SIR.

Although, we already showed there exists an equal-SIR equilibrium in NPG and that each user tries to achieve $\tilde{\gamma}$, we have omitted the possibility that there exists no set of powers in the strategy space where all users achieve the equilibrium SIR. In order to see if such a power vector exists, we examine the conditions under which $\gamma_i = \tilde{\gamma}$ for all i.

5 Equal-SIR Equilibrium

The equal-SIR equilibrium power vector, $\mathbf{p}$, can be derived using the SIR requirement. At equilibrium,

$$\gamma_i = \frac{W}{R} \frac{h_i p_i}{\sum_{j=1, j\neq i}^{n} h_j p_j + \sigma^2} = \tilde{\gamma} \quad \text{for all } i = 1, 2, \cdots, n \quad . \tag{11}$$

In addition to satisfying the above system of equalities, the equilibrium power vector must be nonnegative, $\mathbf{p} \geq 0$. We assume that for the set of powers that solve (11), we have $p_i \leq \overline{p}_i$ for all i. If we substitute $q_i = h_i p_i$ which is the received power of user i at the base station and rearrange terms in equation (11), we get

$$q_i = \frac{R\tilde{\gamma}}{W} \left(\sum_{j=1, j\neq i}^{n} q_j + \sigma^2 \right) \quad . \tag{12}$$

Alternatively, in matrix notation,

$$\mathbf{q} = \frac{R\tilde{\gamma}}{W} (\mathbf{U} - \mathbf{I})\, \mathbf{q} + \frac{R}{W} \tilde{\gamma} \sigma^2 \mathbf{u} \Longrightarrow (\mathbf{I} - \mathbf{A})\mathbf{q} = \frac{R}{W} \tilde{\gamma} \sigma^2 \mathbf{u}, \tag{13}$$

where $\mathbf{U}$ is the $n \times n$ matrix with all 1's, $\mathbf{u}$ is the $1 \times n$ vector with all 1's and $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{A} = \frac{R\tilde{\gamma}}{W} (\mathbf{U} - \mathbf{I})$. There exists a *unique* solution to (13), if $(\mathbf{I} - \mathbf{A})$ is an invertible matrix. The problem can now be stated as follows. Find a vector $\mathbf{q} \geq 0$ such that

$$\mathbf{q} = \frac{R}{W} \tilde{\gamma} \sigma^2 (\mathbf{I} - \mathbf{A})^{-1} \mathbf{u} \quad . \tag{14}$$

Since the right hand side of the above equation is positive, a positive matrix $(\mathbf{I} - \mathbf{A})^{-1}$ will guarantee a positive power vector solution. The matrix $(\mathbf{I} - \mathbf{A})^{-1}$ is nonnegative if the largest eigenvalue $\lambda_{\max}$ of $\mathbf{A}$ is less than 1. By the *Perron-Frobenius Theorem*, the largest eigenvalue $\lambda_{\max}$ of a positive matrix is real and positive, and so are the components of the eigenvector corresponding to $\lambda_{\max}$ [18, 19]. Following an analysis similar to the one given in [2], it can be verified that in our case requiring $\lambda_{\max} < 1$ is equivalent to $\frac{R\tilde{\gamma}}{W}(n-1) < 1$ or

$$n < 1 + \frac{W}{R\tilde{\gamma}} \quad . \tag{15}$$

The equilibrium powers can be found by examining the SIR equations. In (12), add the term $\frac{R\tilde{\gamma}}{W}q_i$ on both sides of the equation to get

$$(1+\frac{R\tilde{\gamma}}{W})q_i = \frac{R\tilde{\gamma}}{W}\left(\sum_{j=1}^{n} q_j + \sigma^2\right) \quad \text{, for all } i = 1, 2, \cdots, n \quad . \tag{16}$$

Observe that the right hand side is independent of user index i and therefore the above expression is the same for all i. We conclude that at equilibrium the received powers of all the users are equal, $h_i p_i = h_j p_j$ for all $i, j \in N$. We rewrite equation (11) to get

$$\tilde{\gamma} = \frac{W}{R}\frac{h_i p_i}{(n-1)h_i p_i + \sigma^2} \quad . \tag{17}$$

Obtaining the equilibrium power p_i is straightforward

$$p_i = \frac{\tilde{\gamma}\sigma^2}{h_i\left(\frac{W}{R} - \tilde{\gamma}(n-1)\right)} \quad . \tag{18}$$

It can be observed that since at the equal-SIR equilibrium $h_i p_i = h_j p_j$ for all $i \neq j$, the powers are ordered, i.e. for $h_i > h_j$ we have $p_i < p_j$. Further, since the equilibrium SIRs are the same and the powers are ordered, the utilities are also ordered where for $h_i > h_j$ we have $u_i > u_j$. Therefore, at the equal-SIR equilibrium, the terminals with better path gain obtain higher utility while expending less power.

Finally, it is important to realize that the equal-SIR analysis holds given that the equilibrium powers given in (18) are such that $0 \leq p_i \leq \overline{p}_i$ for all i in addition to the constraint in (15) being satisfied. If (a) p_i as a solution to (18) exceeds $\overline{p}_i$ for any user i, or (b) the number of users is such that (15) is violated, then at the Nash equilibrium, a group of users will have $p_i = \overline{p}_i$ and the rest of the users will attain $\gamma_i = \tilde{\gamma}$. Unfortunately, no closed form expression can be provided for the equilibrium powers or SIRs in the case when conditions (a) or (b) hold.

6 Pareto Dominant Equilibria

Recall the equal-SIR equilibrium of NPG where utilities are maximized. It is possible to find another power vector which results in utilities at least as good as the Nash equilibrium. Therefore, the NPG equilibrium is not Pareto optimal [4].

Definition 3 *A power vector,* $\hat{\mathbf{p}}$ Pareto dominates *another vector,* $\mathbf{p}$ *if for all* $i \in N$, $u_i(\hat{\mathbf{p}}) \geq u_i(\mathbf{p})$ *and for some* $j \in N$, $u_j(\hat{\mathbf{p}}) > u_j(\mathbf{p})$. *Furthermore, a power vector,* $\mathbf{p}^*$, *is* Pareto optimal, *if there exists no other power vector* $\mathbf{p}$ *such that* $u_i(\mathbf{p}) \geq u_i(\mathbf{p}^*)$ *for all* i *and* $u_j(\mathbf{p}) > u_j(\mathbf{p}^*)$ *for some* j.

We use the terms Pareto dominant and Pareto superior synonymously. Figure 3 explains the concept of Pareto superiority and Pareto optimality on a generic utility possibility set. The attainable values of the utility function for all users constitute the elements of the utility possibility set, $u = \{(u_1, \cdots, u_n) \in$

$\Re^n : u_i \leq u_i(\mathbf{p}),$ for all $i,$ for each $\mathbf{p} \in P\}$. In the example in Figure 3, there are two terminals in the game and their strategy sets are mapped to the utility possibility set shown as the shaded area. Any power vector that provides a Pareto improvement with respect to $\mathbf{y}$ results in non-decreasing changes in individual utilities, $u_i(\mathbf{y})$, and therefore would lie in the area labeled *Region of Pareto improvement*. From the figure, we can observe that $\mathbf{x}$ is such a point. We can also refer to $\mathbf{x}$ as the *Pareto-preferred* power vector when compared to $\mathbf{y}$. The concepts of Pareto-dominance and Pareto-optimality should not be confused: Pareto-optimal power allocations *do not* Pareto-dominate *all* other power vectors.

We now seek improvements to the outcome obtained as a result of the NPG. Power vectors that improve utilities in the Pareto sense are considered dominant with respect to the Nash equilibrium of the NPG. To see how this is possible, suppose each terminal in NPG reduces its equilibrium power by a factor of μ where $0 < \mu \leq 1$. The new power vector is then $\mu\mathbf{p} = (\mu p_1, \cdots, \mu p_n)$. The utility of user i with these reduced powers is

$$u_i(\mu) = \frac{LR}{M\mu p_i} f(\gamma_i^\mu) \tag{19}$$

where

$$\gamma_i^\mu = \frac{W}{R} \frac{\mu h_i p_i}{\sum_{j=1, j\neq i}^{n} \mu h_j p_j + \sigma^2} \quad . \tag{20}$$

We need to examine how the utility value changes for all terminals as the value of μ changes. As the value of μ goes from 1 to 0, the terminals dissipate a power lower than equilibrium powers. If this decrease in μ results in nondecreasing utilities, we have a proof that there exists a power vector that Pareto dominates the Nash equilibrium. Taking the first order derivative of utility in (19) with respect to μ and evaluating the resulting expression at $\mu = 1$ (the Nash equilibrium) we get,

$$\frac{\partial u_i(\mu)}{\partial \mu}|_{\mu=1} = \frac{LR}{Mp_i} f(\gamma_i) \left(\frac{\sigma^2}{\sum_{j=1, j\neq i}^{n} h_j p_j + \sigma^2} - 1 \right) \quad . \tag{21}$$

Notice that the above expression has a negative value, i.e. $\frac{\partial u_i(\mu)}{\partial \mu}|_{\mu=1} < 0$. Therefore, as μ tends to decrease, utilities have a tendency to increase. Since at some $\mu < 1$, the utilities of all the users increase, by definition the Nash equilibrium is not a Pareto optimum. The scalar μ was taken to have the same value for all users for purposes of convenient illustration of the non-Pareto optimality property of the Nash equilibrium. However, when seeking Pareto efficient (Pareto optimal) power vectors, we are not constrained to power vectors that are scalar multiples of the equilibrium power vector. In the rest of the work, we seek to improve the utilities obtained at the Nash equilibrium of NPG.

In improving the equal-SIR equilibrium of NPG in the Pareto sense, pricing schemes emerge as incentive-generating tools that encourage *social* behavior among users. Through pricing, we can increase system performance by implicitly inducing cooperation and yet maintain the noncooperative nature of the game. Typically, pricing serves two purposes: (1) it generates revenue for the system, (2) it encourages players to use system resources more efficiently. In this work we concentrate on the second function of pricing.

By making some observations about the equilibrium of the NPG, it is easier to understand how pricing should be introduced into our problem. At the equal-SIR equilibrium equilibrium, the power $q_i = h_i p_i$ received at the base station is the same for all i. Therefore, at equilibrium the utility is,

$$u_i = \frac{h_i L R f(\tilde{\gamma})}{M q_i} \quad . \tag{22}$$

Since $\tilde{\gamma}$ and q_i are the same for all users, utilities at equilibrium can be ordered based on their path gain: if $h_i > h_j$, then at equilibrium $u_i > u_j$. In the absence of shadow fading path loss decreases with the terminal distance. Therefore, we conclude users closer to the base station on average achieve higher utilities at the NPG equilibrium. If we index the terminals in order of increasing distance form the base station, where the distance of terminal i is d_i meters, we have at equilibrium (for $d_1 < d_2 < \cdots < d_n$)

$$\begin{aligned} u_1 > u_2 > \cdots > u_n \\ p_1 < p_2 < \cdots < p_n \end{aligned} \tag{23}$$

Earlier in this section we showed that the strategy of maximizing utility leads everyone to transmit at a power that is too high. Thus, we seek a means to encourage terminals to transmit at lower power. To derive such a technique, we examine the effect of each terminal's power adjustment on the utility of all other terminals. The *harm* on terminal j of a power adjustment at terminal i is given by the *harm coefficient* [4, 14] and defined as,

$$C_{ij} = -\frac{\partial u_j}{\partial p_i} p_i \quad \text{bits/Joule} \quad \text{for all } i \neq j. \tag{24}$$

The total harm imposed on all terminals by terminal i transmitting at a power level p_i is,

$$C_i = \sum_{j=1, j\neq i}^{n} C_{ij} \quad \text{bits/Joule} \quad . \tag{25}$$

For the systems given in Table 1, we have discovered that at equilibrium, the harm imposed by each terminal is a monotonic increasing function of the distance of the terminal from the base station. Therefore, for terminals indexed in order of increasing distance from the base station, we have at equilibrium,

$$C_1 < C_2 < \cdots < C_n \tag{26}$$

Since by examining (26) and (23), it can be observed that terminals that transmit higher power at equilibrium cause higher harm to the system, it is reasonable to impose a price on each transmission that increases monotonically with power [4].

7 Non-cooperative Power Control with Pricing

Keeping the above guidelines for a pricing strategy in mind, we develop a non-cooperative game with pricing. Let $\Gamma_c = [N, \{P_i\}, \{u_i^c(\cdot)\}]$ denote an n-player noncooperative power-control game with pricing (NPGP). Utilities for NPGP are

$$u_i^c(\mathbf{p}) = u_i(\mathbf{p}) - c_i(p_i, \mathbf{p}_{-i}) \tag{27}$$

where $c_i : P \to \Re_+^1$ is the pricing function for all $i \in N$. The multi-objective optimization problem that NPGP solves can be expressed as,

$$\begin{aligned} \textbf{(NPGP)} \quad \max_{p_i} \quad & u_i^c(p_i, \mathbf{p}_{-i}) = u_i(\mathbf{p}) - c_i(p_i, \mathbf{p}_{-i}), \quad \text{for all } i = 1, 2, \cdots, n \\ \text{s.t.} \quad & p_i \in P_i, \quad \text{for all } i = 1, 2, \cdots, n. \end{aligned} \tag{28}$$

Note that there is no restriction on the form of the pricing function $c_i(\cdot)$. However, considering that at equilibrium the terminal that transmits more power causes more harm to the system, we impose a price that increases with the transmit power of the user. In this work, we restrict our attention to linear pricing schemes (see also [4]) of the form

$$c_i(p_i, \mathbf{p}_{-i}) = c\alpha_i p_i, \tag{29}$$

where c and $\{\alpha_i\}$ are positive scalars[1].

The pricing factor, c, needs to be tuned such that user self-interest leads to best possible improvement in overall network performance. The NPGP with linear pricing is as follows.

$$\begin{aligned} \textbf{(NPGP)} \quad \max_{p_i} \quad & u_i(\mathbf{p}) - c\alpha_i p_i, \quad \text{for all } i = 1, \cdots, n \\ \text{s.t.} \quad & p_i \in P_i, \quad \text{for all } i = 1, \cdots, n. \end{aligned} \tag{30}$$

Notice that the NPGP is practically the same game as the NPG with different payoff functions. We seek a Nash equilibrium point that solves NPGP, if one exist. In game $\Gamma = [N, \{P_i\}, \{u_i(\cdot)\}]$, user utilities are quasiconcave in their own strategy. We established that in a game with such utility functions there exists an equilibrium. The NPGP, however, does not have quasiconcave utility functions. Analytical techniques used to prove Nash existence under strong assumptions of convexity and differentiability are no longer applicable. Thus, we turn to supermodularity theory to show existence of equilibria.

We now consider the theory of supermodular games to investigate the properties of equilibria in the NPGP.

[1] The pricing factor, c, has units $b/s/W^2$ such that it is consistent with the units of the net utility u_i^c in b/J.

7.1 Supermodular Games and NPGP

Supermodularity was introduced into the game theory literature by Topkis [20] in 1979. In a supermodular power control game, each player's desire to increase its power increases with an increase in other players' powers. Supermodular games are of particular interest since they have Nash equilibria, and there exists an upper and a lower bound on Nash strategies of each user [10]. Convexity assumptions are no longer needed for Nash equilibrium existence proofs.

Definition 4 *Consider a game* $\Gamma = [N, \{P_i\}, \{u_i(\cdot)\}]$ *with strategy spaces* $P_i \subset \mathcal{R}^{m_i}$ *for all* i*.* Γ *is* supermodular *if for each* i*,* P_i *is a sublattice*[2] *of* $\mathcal{R}^{m_i}$*, and* $u_i(p_i, \mathbf{p}_{-i})$ *has non-decreasing differences (NDD) in* $(p_i, \mathbf{p}_{-i})$ *and* $u_i(p_i, \mathbf{p}_{-i})$ *is supermodular in* p_i[3]*.*

Since P_i is a single dimensional set, supermodularity in p_i is guaranteed. Also, in our work, since $\mathcal{R}^{m_i} = \mathcal{R}$ and since P_i is taken as a convex and compact subset bounded by a minimum and a maximum power constraint, P_i is a sublattice of $\mathcal{R}$. Non-decreasing differences describe the complementarity among the players' strategies.

Definition 5 $u_i(p_i, \mathbf{p}_{-i})$ *has NDD in* $(p_i, \mathbf{p}_{-i})$ *if for all* $\mathbf{p}_{-i} \geq \mathbf{p}'_{-i}$ *the quantity* $u_i(p_i, \mathbf{p}_{-i}) - u_i(p_i, \mathbf{p}'_{-i})$ *is nondecreasing in* p_i*. Equivalently,* $u_i(p_i, \mathbf{p}_{-i})$ *has NDD in* $(p_i, \mathbf{p}_{-i})$ *iff* $\frac{\partial^2 u_i(\mathbf{p})}{\partial p_i \partial p_j} \geq 0$ *for all* $j \neq i$*.*

If the utility of user i has NDD in the power vector, user i increases its utility if it increases its power level in response to an increase in the power level of another user j. The significance of this property is rooted in the fact that such utilities lead to a system of best response correspondences that have a fixed point [21, p. 180]. A fixed point in best response correspondences implies a Nash equilibrium.

Theorem 3 *[21] The set of Nash equilibria of a supermodular game, denoted by* E *is non-empty. Furthermore,* E *has a largest element and a smallest element, denoted by* $\mathbf{p}_{\max}$ *and* $\mathbf{p}_{\min}$*, respectively.*

A proof of the theorem can be found in [21]. The largest and smallest vector in a set of vectors refer to the component-wise comparison between vectors in that set. More formally, $\mathbf{p}_{\min} \leq \mathbf{p} \leq \mathbf{p}_{\max}$ for all $\mathbf{p} \in E$. Figure 4 shows a generic strategy space P for a 2-player game where E is the subset of P that contains all the Nash equilibria of the game. Consider a game with exogenous parameter, ϵ, $\Gamma_\epsilon = [N, \{P_i\}, \{u_i^\epsilon(\cdot)\}]$ with utilities $u_i(p_i, \mathbf{p}_{-i}, \epsilon)$.

Definition 6 *A game with an exogenous parameter is said to be* supermodular*, or it is a* parameterized game with complementarities *if* $u_i(p_i, \mathbf{p}_{-i}, \epsilon)$ *has NDD in* $(p_i, \mathbf{p}_{-i})$ *and in* (p_i, ϵ) *for all* i*.*

The next result follows directly.

Theorem 4 *In a parameterized supermodular game, both* $\mathbf{p}_{\min}(\epsilon)$ *and* $\mathbf{p}_{\max}(\epsilon)$ *are nondecreasing in* ϵ*.*

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[2] *P_i is a sublattice of $\mathcal{R}^{m_i}$ if $x \in P_i$ and $y \in P_i$ imply that $x \vee y \in P_i$ and $x \wedge y \in P_i$ where* $x \vee y = (\max(x_1, y_1), \cdots, \max(x_{m_i}, y_{m_i}))$ *and* $x \wedge y = (\min(x_1, y_1), \cdots, \min(x_{m_i}, y_{m_i}))$*.*

[3] *Utility function $u_i(p_i, \mathbf{p}_{-i})$ is supermodular in p_i if for each $\mathbf{p}_{-i}$ we have* $u_i(p_i, \mathbf{p}_{-i}) + u_i(p'_i, \mathbf{p}_{-i}) \leq u_i(p_i \vee p'_i, \mathbf{p}_{-i}) + u_i(p_i \wedge p'_i, \mathbf{p}_{-i})$ *for all* $p'_i \in P_i$*.*

In a parameterized game the Nash set E *moves* as the parameter is changed. Figure 4 illustrates how $\mathbf{p}_{\min}(\epsilon)$ and $\mathbf{p}_{\max}(\epsilon)$ change within the two dimensional strategy space as ϵ is increased. The proof can be found in [21].

NPGP $\Gamma_c = [N, \{P_i\}, \{u_i^c(\cdot)\}]$ is not a supermodular game by Definition 6. However, if the strategy spaces of users are modified appropriately, we can show that the resulting game is supermodular. The modified strategy space for user i denoted by $\hat{P}_i$ is a compact set defined by $\hat{P}_i = [\underline{p}_i, \overline{p}_i]$ where the smallest power in the strategy set $\underline{p}_i$ is derived from $\underline{\gamma_i} \geq 2 \ln M$. We find this SIR requirement using the condition given in Definition 5, i.e. $\frac{\partial^2 u_i(\mathbf{p})}{\partial p_i \partial p_j} \geq 0$ for all $j \neq i$. The largest power $\overline{p}_i$ is the maximum power constraint of the system. In this work, we assume the modified strategy space $\hat{P}$, which is the Cartesian product of individual strategy spaces, $\{\hat{P}_i\}$, is non-empty, i.e. there exists a $\underline{p}_i$ such that $0 < \underline{p}_i \leq \overline{p}_i$ for all i. Note that in the modified strategy space of NPGP described above, the power levels that yield $\gamma_i \leq 2 \ln M$ are no longer available to the terminal. In NPG, the terminals can use any nonnegative power as long as it is below the maximum power limit. Thus, the users in NPGP operate in a smaller feasibility region as compared to the terminals in NPG. Figure 4 shows $\underline{\mathbf{p}}$ and $\overline{\mathbf{p}}$ in a generic two dimensional strategy space.

Theorem 5 *Modified NPGP $\hat{\Gamma}_c = [N, \{\hat{P}_i\}, \{u_i^c(\cdot)\}]$ with exogenous parameter c is a supermodular game.*

Proof: We test whether the conditions in Definition 6 are satisfied. $\hat{\Gamma}_c$ has NDD in $(p_i, \mathbf{p}_{-i})$ since the condition given in Definition 5 yields the same expression for $\hat{\Gamma}_c$ as $\hat{\Gamma}$. We need only check whether the utility $u_i(p_i, \mathbf{p}_{-i}, c)$ has NDD in (p_i, c). First, perform a change of variables from c to ϵ where $\epsilon = -c$. When we take the partial derivative with respect to both p_i and ϵ, we get $\frac{\partial u_i}{\partial p_i \partial \epsilon} = \alpha_i \geq 0$ for all i. Thus, $\hat{\Gamma}_c$ is supermodular. ■

Using Theorems 3 and 4, we have all the Nash equilibria of $\hat{\Gamma}_c$ within the set $E_c = \{\mathbf{p} \in \hat{P} : \hat{\mathbf{p}}_{\min}(c) \leq \mathbf{p} \leq \hat{\mathbf{p}}_{\max}(c)\}$, and both $\hat{\mathbf{p}}_{\min}(c)$ and $\hat{\mathbf{p}}_{\max}(c)$ are nonincreasing in c. Although we have shown that an equilibrium exists for modified NPGP, we are yet to show how that point is reached. Let $\underline{\mathbf{p}}$ and $\overline{\mathbf{p}}$ denote the smallest and the largest vectors in modified strategy space $\hat{P}$, respectively. Consider the following algorithm.

Algorithm 1 *Starting from $\underline{\mathbf{p}}$ (or $\overline{\mathbf{p}}$), update the power vector in a round-robin fashion where each user maximizes $u_i^c(\mathbf{p})$ taking turns.*

The following result states the outcome of the algorithm given above.

Theorem 6 *The sequence of power vectors generated by Algorithm 1 starting from $\underline{\mathbf{p}}$ (or $\overline{\mathbf{p}}$) converges to $\hat{\mathbf{p}}_{\min}(c)$ (or $\hat{\mathbf{p}}_{\max}(c)$).*

The proof of the above theorem can be shown using arguments similar to the one used in [21]. Experiments suggest $\hat{\mathbf{p}}_{\min}(c) = \hat{\mathbf{p}}_{\max}(c)$ for our problem. If this is indeed true analytically, it implies that the Nash equilibrium in the modified NPGP is unique and can be reached from either the *top* or the *bottom* of the strategy space by implementing Algorithm 1. Since we do not know if there is a unique equilibrium, we compare the equilibria in the Nash set E_c to determine if there exists a single equilibrium that dominates all other equilibria. Indeed, we can show that $\hat{\mathbf{p}}_{\min}(c)$ is the *best* equilibrium in the set E_c.

Theorem 7 *If $\mathbf{x}, \mathbf{y} \in E_c$ are two Nash equilibria in modified NPGP where $\mathbf{x} \geq \mathbf{y}$, then $u_i^c(\mathbf{x}) \leq u_i^c(\mathbf{y})$ for all i.*

Proof: Notice that for fixed p_i and c, utility $u_i^c = \frac{LR}{Mp_i} f(\gamma_i) - c\alpha_i p_i$ decreases with increasing $\mathbf{p}_{-i}$ for all i. Therefore, since $\mathbf{x}_{-i} \geq \mathbf{y}_{-i}$ we have,

$$u_i^c(x_i, \mathbf{x}_{-i}) \leq u_i^c(x_i, \mathbf{y}_{-i}). \tag{31}$$

Also, by definition of Nash equilibrium and since $\mathbf{y}$ is a Nash equilibrium of NPGP, we have

$$u_i^c(x_i, \mathbf{y}_{-i}) \leq u_i^c(y_i, \mathbf{y}_{-i}). \tag{32}$$

By the above equations,

$$u_i^c(\mathbf{x}) \leq u_i^c(\mathbf{y}). \tag{33}$$

■

Corollary 1 *For modified NPGP, $\hat{\mathbf{p}}_{\min}(c) \in E_c$ is the Pareto-dominant equilibrium, i.e. $u_i^c(\hat{\mathbf{p}}_{\min}(c)) \geq u_i^c(\mathbf{p}^c)$ for all i, for all $\mathbf{p}^c \in E_c$*

Proof: By Theorem 7 we know that componentwise smaller equilibrium results in higher utilities for all users than a larger equilibrium. Since $\hat{\mathbf{p}}_{\min}(c) \leq \mathbf{p}^c$ for all $\mathbf{p}^c \in E_c$, we conclude that for all $\mathbf{p}^c \in E_c$,

$$u_i^c(\hat{\mathbf{p}}_{\min}(c)) \geq u_i^c(\mathbf{p}) \tag{34}$$

for all i. ■

Note that this result implies that in the presence of multiple Nash equilibria, the one that yields highest utilities is the Nash equilibrium with the minimum total transmit powers.

The rest of the work is devoted to searching for an NPGP equilibrium that brings maximal Pareto improvement to the equal-SIR equilibrium in NPG. The value of the pricing factor that results in maximal Pareto improvement is defined as follows. As we increase the value of c and calculate equilibrium power vectors and corresponding utilities, we observe that utilities increase with c. However, at some point, at least one user begins to receive decreasing utilities at equilibrium with higher values of c. We declare it the point of maximal Pareto improvement and denote the corresponding value of the pricing factor by c_{best}. The improvement in utilities with pricing at c_{best} as compared to the utilities obtained in NPG are presented in section 8.

8 Numerical Results

We demonstrate the improvement in performance obtained as a result of the NPGP outcome on a single-cell CDMA system with stationary users, fixed frame size and no forward error correction. The system we examine has the design parameters listed in Table 2. Also, the system we consider has 9 terminals that are located at $\mathbf{d} = [310, 460, 570, 660, 740, 810, 880, 940, 1000]$ meters from the base station. Path gains are obtained using the simple path loss model $h_i = K/d_i^4$ where $K = 0.097$ is a constant.

For this system, the efficiency function is as given in equation (7). Using this efficiency function and the linear pricing regime with $\alpha_i = 1$ for all i, the equilibrium powers that solve the NPGP given in (30) are obtained by use of Algorithm 1. We first get the equilibrium powers in NPGP with no pricing ($c = 0$), which is equivalent to playing the NPG given in (8). Recall that the equilibrium powers in NPG are obtained by solving $\gamma_i = \tilde{\gamma}$ for all i. The equilibrium SIR for the specific system under examination is found to be $\tilde{\gamma} = 12.4$ by solving $f'(\gamma)\gamma = f(\gamma)$ or (44). When we substitute $\tilde{\gamma}$ in equation (15), we obtain the feasibility condition for this system as $n < 9.05$. Once the equilibrium with no pricing is obtained, the NPGP is played again after incrementing the pricing factor, c, to a positive value. Algorithm 1 returns a set of powers at equilibrium with this value of the pricing factor. If the utilities at this new equilibrium with some positive price c improve with respect to the previous instance, the pricing factor is incremented and the procedure is repeated. We continue until an increase in c results in utility levels worse than the previous equilibrium values for at least one user. We declare the last value of c with Pareto improvement to be the best pricing factor, c_{best}. Figure 5 is constructed by letting Algorithm 1 reach Nash equilibrium at each value of c. We terminate incrementing the pricing factor if at least one user receives worse payoff than the previous instance of the game. It can be observed that solution by NPGP with $c = c_{\text{best}}$ offers a significant improvement in total utilities with respect to the solution offered by NPG. Increase in individual utilities can be examined in Figure 6. The corresponding equilibrium powers are displayed in Figure 7. The terminals that are closer to the base station receive much higher utilities while expending smaller power as compared to terminals further away from the base station in both NPG and NPGP equilibria. However, we observe that utilities improve significantly as a result of pricing while the powers decrease from values at equilibrium with no pricing. The numerical results also reveal that although the equilibrium SIRs for the game with zero pricing are equal for all terminals, the SIRs at equilibrium in NPGP with $c = c_{\text{best}}$ are higher for terminals closer to the base station.

M, total number of bits per frame	80
L, number of information bits per frame	64
W, spread spectrum bandwidth	10^6 Hz
R, bit rate	10^4 bits/second
σ^2, AWGN power at the receiver	5×10^{-15} Watts
modulation technique	non-coherent FSK
$\overline{p}$, maximum power constraint	2 Watts

Table 2: The list of parameters for the single-cell CDMA system used in the experiments.

9 NPGP and the Social Optimum

In NPGP, we choose the value of $c = c_{\text{best}}$ that brings maximal Pareto improvement to the solution from NPG. However, the power vector obtained as a result is not necessarily a social optimum. In this section, we discuss the connection between a social optimum and a general pricing function. The pricing function is not restricted to have a linear form.

Theorem 8 *A power vector $\mathbf{p}^*(\boldsymbol{\beta})$ that solves the social problem $(S_{\boldsymbol{\beta}})$ is Pareto optimal where $(S_{\boldsymbol{\beta}})$ is defined as*

$$(S_{\boldsymbol{\beta}}) \qquad \max_{\mathbf{p}} \boldsymbol{\beta} \cdot \mathbf{u} = \max_{\mathbf{p}} \sum_{i=1}^{n} \beta_i u_i \tag{35}$$

with $\boldsymbol{\beta}$ a vector of positive scalars.

Proof: Assume $\mathbf{p}^*(\boldsymbol{\beta})$ solves $(\mathrm{S}_{\boldsymbol{\beta}})$ and suppose $\mathbf{p}^*(\boldsymbol{\beta})$ is not Pareto optimal. Then, there exists some power vector $\mathbf{p}'$ such that $u_i(\mathbf{p}') > u_i(\mathbf{p}^*)$ for some i and $u_i(\mathbf{p}') \geq u_i(\mathbf{p}^*)$ for all i. This implies that $\sum_{i=1}^{n} \beta_i u_i(\mathbf{p}') > \sum_{i=1}^{n} \beta_i u_i(\mathbf{p}^*)$. Therefore $\mathbf{p}^*(\boldsymbol{\beta})$ cannot be a solution to $(\mathrm{S}_{\boldsymbol{\beta}})$ which is a contradiction to the original assumption. Therefore, it has to be a Pareto optimal point. ■

In fact, in our experiments, we observe that the sum of utilities continue to increase beyond $c = c_{\text{best}}$. However, such improvement in total utilities result in degraded QoS for at least one user, beginning with the user that is farthest from the base station. The solution to $(\mathrm{S}_{\boldsymbol{\beta}})$ is not even guaranteed to Pareto dominate the Nash solution of the NPG.

Solving $(\mathrm{S}_{\boldsymbol{\beta}})$ with a particular choice of $\boldsymbol{\beta}$ results in one of the points in the Pareto-optimal frontier, which consists of the points on the north-east boundary of the utility possibility set as shown in Figure 3. By solving $(\mathrm{S}_{\boldsymbol{\beta}})$ for all $\boldsymbol{\beta} \in \Re_+^n$ we can construct the Pareto-optimal frontier. What we obtain by the NPGP is a Pareto dominant power vector with respect to the solution offered by the NPG. NPGP solution lies in the space labeled *Region of Pareto improvement* in Figure 3. Note that unless the utility possibility set is a convex set, solution of the social optimum is not guaranteed to yield a point that is Pareto superior to the NPG.

Nevertheless, an optimal pricing function that has the solution of the social problem as a Nash equilibrium does exist for each user.

Theorem 9 *Let $\mathbf{p}^*(\boldsymbol{\beta})$ solve the social problem $(S_{\boldsymbol{\beta}})$. $\mathbf{p}^*$ is also a Nash equilibrium for the NPGP given in (28) with pricing function $c_i(\mathbf{p}) = -\frac{1}{\beta_i} \sum_{j=1, j\neq i}^{n} \beta_j u_j(\mathbf{p})$.*

Proof: $\mathbf{p}^*$ is a Nash equilibrium of the NPGP if $u_i^c(\mathbf{p}_i^*, \mathbf{p}_{-i}^*) \geq u_i^c(p_i', \mathbf{p}_{-i}^*)$ for all $i \in N$, for all $p_i' \in P_i$. Since $\mathbf{p}^*$ solves the social problem,

$$\sum_{j=1}^{n} \beta_j u_j(\mathbf{p}^*) \quad \geq \quad \sum_{j=1}^{n} \beta_j u_j(p_i', \mathbf{p}_{-i}^*) \tag{36}$$

for all $(p_i', \mathbf{p}_{-i}^*) \in P$. Rearranging terms on both sides,

$$\beta_i u_i(\mathbf{p}^*) + \sum_{j=1, j\neq i}^{n} \beta_j u_j(\mathbf{p}^*) \quad \geq \quad \beta_i u_i(p_i', \mathbf{p}_{-i}^*) + \sum_{j=1, j\neq i}^{n} \beta_j u_j(p_i', \mathbf{p}_{-i}^*) \tag{37}$$

Since it is a positive constant, we can divide both sides by β_i while preserving the direction of the inequality,

$$u_i(\mathbf{p}^*) + \frac{1}{\beta_i} \sum_{j=1, j\neq i}^{n} \beta_j u_j(\mathbf{p}^*) \quad \geq \quad u_i(p_i', \mathbf{p}_{-i}^*) + \frac{1}{\beta_i} \sum_{j=1, j\neq i}^{n} \beta_j u_j(p_i', \mathbf{p}_{-i}^*) \tag{38}$$

Consider the second term on either side as the cost function and let $c_i(\mathbf{p}) = \frac{1}{\beta_i} \sum_{j=1, j\neq i}^{n} \beta_j u_j(\mathbf{p})$. Expressing in terms of the cost function we get,

$$u_i(\mathbf{p}^*) - c_i(\mathbf{p}^*) \quad \geq \quad u_i(p_i', \mathbf{p}_{-i}^*) - c_i(p_i', \mathbf{p}_{-i}^*) \tag{39}$$

which is true for all $p_i' \in P_i$ and true for all i. This is the definition of a Nash equilibrium. Thus, by definition, $\mathbf{p}^*$ is a Nash equilibrium for the NPGP game. Since the Nash equilibrium of the NPGP game with

$$c_i(\mathbf{p}) = -\frac{1}{\beta_i} \sum_{j=1, j\neq i}^{n} \beta_j u_j(\mathbf{p}) \tag{40}$$

is the point where the social problem is solved, we refer to this pricing function as the Pareto optimal pricing function. ■

Notice that with pricing function (40), each user is trying to maximize the same objective function, individually. The optimal pricing function given here is in the most general form a pricing function can take and does not have the linear form we produced results for in the preceding sections. This pricing function is not practical since we need an algorithm that would guarantee the solution of (35) to emerge as an equilibrium of the social game. However, accomplishing this is as difficult as (if not harder than) the central authority solving the social problem and imposing it on all users. Instead, the implementation proposed in this work has a single pricing factor, c, to be announced by the base station. Thus, users can still implement their distributed power control schemes that unilaterally maximize the utility function $u_i^c(\mathbf{p})$ in equation (27).

10 Summary and Conclusions

We have presented a distributed power control algorithm for wireless data systems. The QoS a wireless terminal receives is referred to as the *utility* and distributed power control where users maximize their utilities is a *non-cooperative power control game* (NPG). The resulting operating point (Nash equilibrium) of such a distributed power control is inefficient in power usage. Therefore, we introduce pricing to improve the NPG result. In the *non-cooperative power control game with pricing* (NPGP), each terminal maximizes its net utility given by the difference between the utility function and a pricing function. The class of pricing functions studied in this work is linear in transmit power, where the pricing function is simply the product of a pricing factor and the transmit power. Such a pricing function allows easy implementation: The power control algorithm is realized by the base station announcing the pricing factor to all the users, which is followed by each terminal choosing the transmit power from its strategy space that maximizes its net utility. For positive values of the pricing factor, we show that there exist Nash equilibria that are not necessarily unique. However, we have proved that the minimum power vector in the set of Nash equilibria yields higher utilities for each user than any other equilibrium power vector. Such a power vector is said to *Pareto dominate* other equilibrium power vectors and it is an obvious choice for operation. We have also presented an algorithm that reaches the Pareto dominant equilibrium starting from the smallest power vector in the strategy space.

Under zero pricing, the users reach an equilibrium set of powers where each terminal receives a common target SIR, $\tilde{\gamma}$. The value of $\tilde{\gamma}$ is determined by the system characteristics such as modulation technique, channel model and packet length. As the pricing factor is increased from zero to positive values, the equilibrium begins to shift towards a point where users attain lower SIR, expend lower power and attain higher utilities. At the equilibrium of NPGP, SIRs are no longer equal for all users. In fact, the equilibrium SIR for a user closer to the base station is higher than a user farther away, while all of the SIRs are smaller than the no-pricing equilibrium of $\tilde{\gamma}$. As a special case of an appropriate choice of the pricing factor, we define c_{best} as the value of the pricing factor where the utility of at least one terminal begins to decrease with increasing values of c. Using $c = c_{\text{best}}$, it is possible to get significant improvement in utility for all terminals. Therefore, we declare this point to be our best utility result. Finally, we have discussed how the utilities obtained using the pricing factor, c_{best}, compare with the social optimum which is the power vector that maximizes the sum of utilities of all the terminals in the system. Our results indicate that linear pricing while yielding Pareto improvements (over the case of no pricing) is still unable to achieve the social optimum.

A Proof of Theorem 1

The set of maximizers of the continuous function $u_i(\cdot, \mathbf{p}_{-i})$ on the compact set P_i in NPG is called the *best-response correspondence* and is denoted by $r_i(\mathbf{p}_{-i})$. It is the mapping $r_i : P_{-i} \rightarrow P_i$ and defined as

$$r_i(\mathbf{p}_{-i}) = \{p_i \in P_i : u_i(p_i, \mathbf{p}_{-i}) \geq u_i(p_i', \mathbf{p}_{-i}) \quad \forall p_i' \in P_i\} \quad . \tag{41}$$

An alternative definition for the Nash equilibrium can be stated using the set of best responses. A power vector $\mathbf{p}$ is a Nash equilibrium of NPG if and only if $p_i \in r_i(\mathbf{p}_{-i})$ for all $i \in N$. When the conditions in Theorem 1 are satisfied, the correspondence $r_i(\cdot)$ is nonempty, convex-valued and upper semicontinuous for all i [22–24]. Thus, there exists a fixed point $\mathbf{p}$ such that $p_i \in r_i(\mathbf{p}_{-i})$ for all $i \in N$. This fixed point is by definition a Nash equilibrium. The proof of the theorem is completed by showing the conditions given in the theorem are met in NPG. Each user has a strategy space that is defined by a minimum power, a maximum power and all the power values in between. We also assume the maximum power is larger than or equal to the minimum power. Thus, the first condition is satisfied. It remains to show that the utility function $u_i(\mathbf{p})$ is quasiconcave in p_i for all i in NPG. First, we define quasiconcavity.

Definition 7 *The function $u_i : P_i \rightarrow \Re^1_+$ defined on the convex set P_i is quasiconcave in p_i if and only if*

$$u_i(\lambda p_i + (1-\lambda)p'_i, \mathbf{p}_{-i}) \geq \min(u_i(p_i, \mathbf{p}_{-i}), u_i(p'_i, \mathbf{p}_{-i})) \tag{42}$$

for all $p_i, p'_i \in P_i$ and $\lambda \in [0,1]$.

Alternatively, either the local maximum of the quasiconcave function is at the same time a global maximum or the quasiconcave function is constant in the neighborhood of a local maximum [25, 26]. We can show that the first part of this condition is true for the utility function used in this study.

For a differentiable function, the first order necessary optimality condition is given as $\frac{\partial u_i(p_i,\mathbf{p}_{-i})}{\partial p_i} = 0$. The partial derivative of $u_i(\cdot)$ with respect to p_i is

$$\frac{\partial u_i(p_i, \mathbf{p}_{-i})}{\partial p_i} = \frac{LR}{Mp_i^2}\left(f'(\gamma_i)\gamma_i - f(\gamma_i)\right) \tag{43}$$

where $f'(\gamma_i) = df(\gamma_i)/d\gamma_i$. Since $p_i \geq 0$ for NPG, we examine only positive real numbers. Evaluating equation (43) at $p_i = 0$, we get $\frac{\partial u_i(p_i,\mathbf{p}_{-i})}{\partial p_i} = 0$. Therefore, $p_i = 0$ is a stationary point and the value of utility at this point is $u_i(0, \mathbf{p}_{-i}) = 0$. If we evaluate utility in the ϵ-neighborhood of $p_i = 0$, where ϵ is a small positive number, we notice that utility is positive which implies utility is increasing at $p_i = 0$. Therefore, we conclude zero can not be a local maximum. For nonzero values of the power, we examine the values of γ_i that make $f'(\gamma_i)\gamma_i - f(\gamma_i) = 0$. Expressing $f'(\gamma_i)$ in terms of $f(\gamma_i)$ and rearranging terms, we get $\frac{M}{2}\gamma_i e^{-\gamma_i/2} - (1 - e^{-\gamma_i/2}) = 0$ or

$$\frac{M}{2}\gamma_i + 1 = e^{\gamma_i/2} \quad . \tag{44}$$

We observe that the right hand side of the above equation is convex in γ_i, the left hand side is monotonously increasing in γ_i, and the equation is satisfied at $\gamma_i = 0$. Therefore, there is a single value that satisfies the given expression for $\gamma_i > 0$. Let this value be $\gamma_i = \tilde{\gamma}$ where $\tilde{\gamma}$ is derived numerically from equation (44) and it is the same value for all users assuming each user operates with the same efficiency function. The second order partial derivative of the utility with respect to the power reveals that this point is a local maximum and therefore a global maximum. Hence, the utility function of user i is quasiconcave in p_i for all i. This completes the proof of the theorem.

B Proof of Theorem 2

In the proof of Theorem 1 in Appendix A, we demonstrated that the unique maximizer for user i in NPG is the power that satisfies $\gamma_i = \tilde{\gamma}$ where $\tilde{\gamma}$ is determined as a result of system parameters. Since we assume that the equilibrium powers that satisfy the given set of equations are all less than or equal to the maximum power constraint, the best response correspondence for user i is given as

$$r_i(\mathbf{p}_{-i}) = \frac{\tilde{\gamma}}{\frac{W}{R} h_i} \left(\sum_{j=1, j \neq i}^{n} h_j p_j + \sigma^2 \right) \qquad \text{for all } i = 1, 2, \cdots, n \tag{45}$$

By definition, if $\mathbf{p}^*$ is a Nash equilibrium, then $p_i^* = r_i(\mathbf{p}_{-i}^*)$ for all i. The power vector $\mathbf{p}^*$ satisfying this expression is basically the power vector that solves (13). Recall from section 5 that if a solution exists, it is unique and is given by (18) for all i. Since, $p_i^* = r_i(\mathbf{p}_{-i}^*)$ is satisfied, it is a Nash equilibrium. Furthermore, at this equilibrium the SIRs are $\tilde{\gamma}$ for all users.

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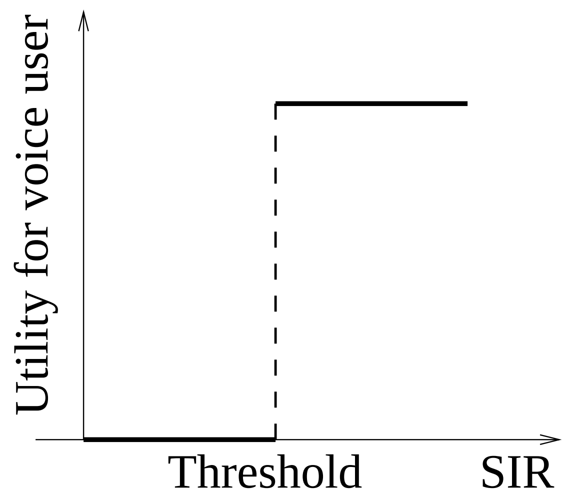

Figure 1: Utility as a function of SIR in voice communications.

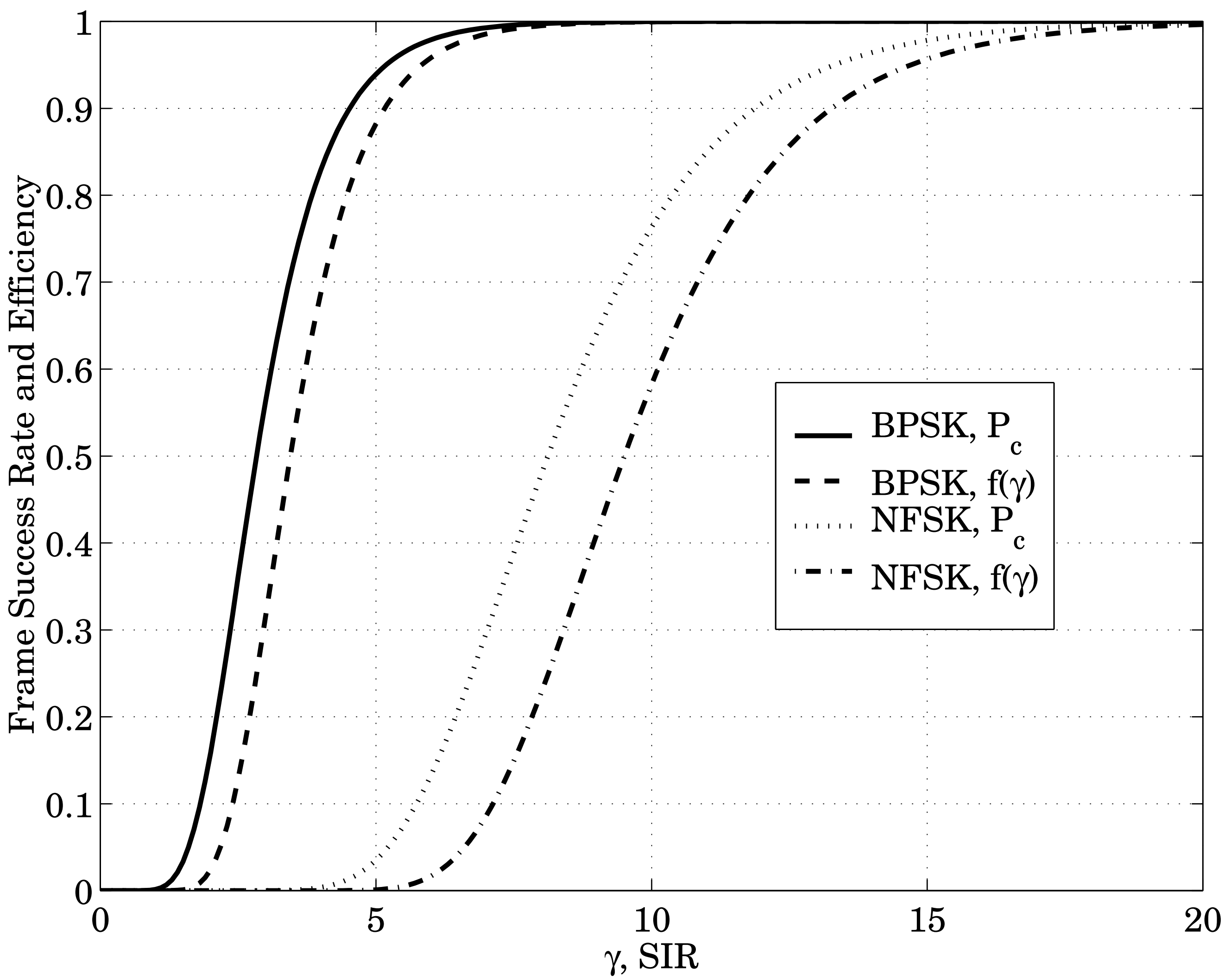

Figure 2: The frame success rate (FSR) and efficiency as a function of terminal SIR for BPSK and non-coherent FSK modulation schemes. Efficiency is an approximation to FSR.

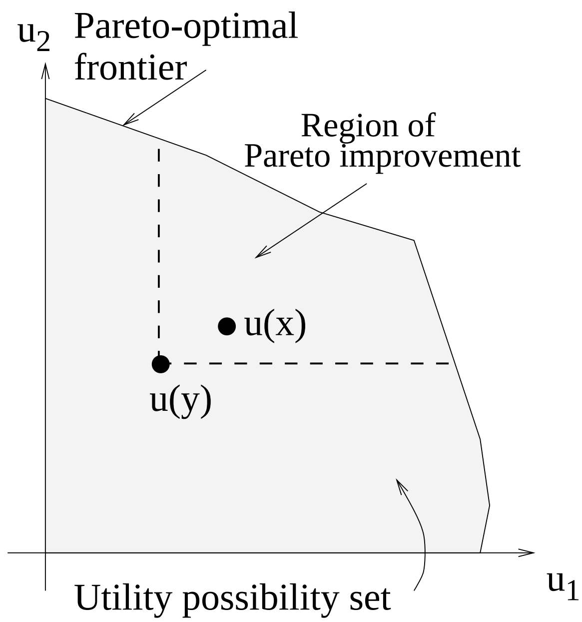

Figure 3: Power vector $\mathbf{x}$ Pareto dominates power vector $\mathbf{y}$.

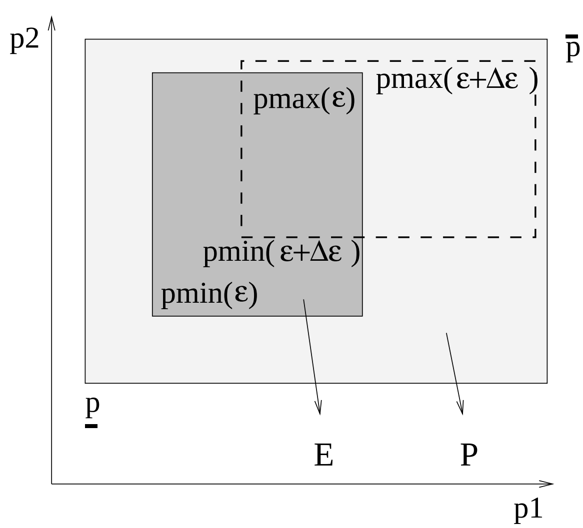

Figure 4: Strategy space (P) and set of Nash equilibria (E) of a supermodular game. If the game is parameterized, $\mathbf{p}_{\min}(\epsilon)$ and $\mathbf{p}_{\max}(\epsilon)$ are nondecreasing in ϵ

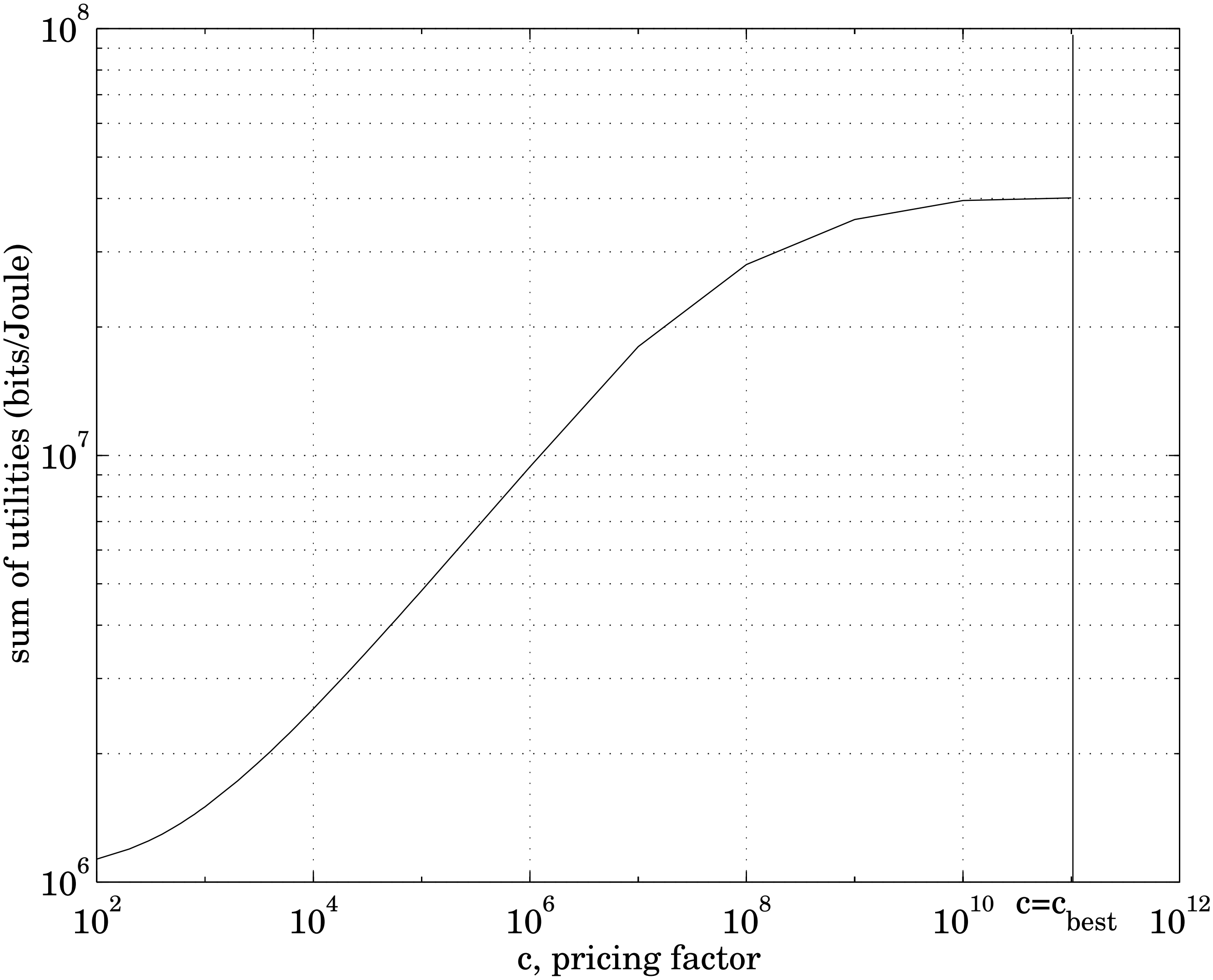

Figure 5: Sum of equilibrium utilities in a game with 9 terminals as a function of the pricing factor, c.

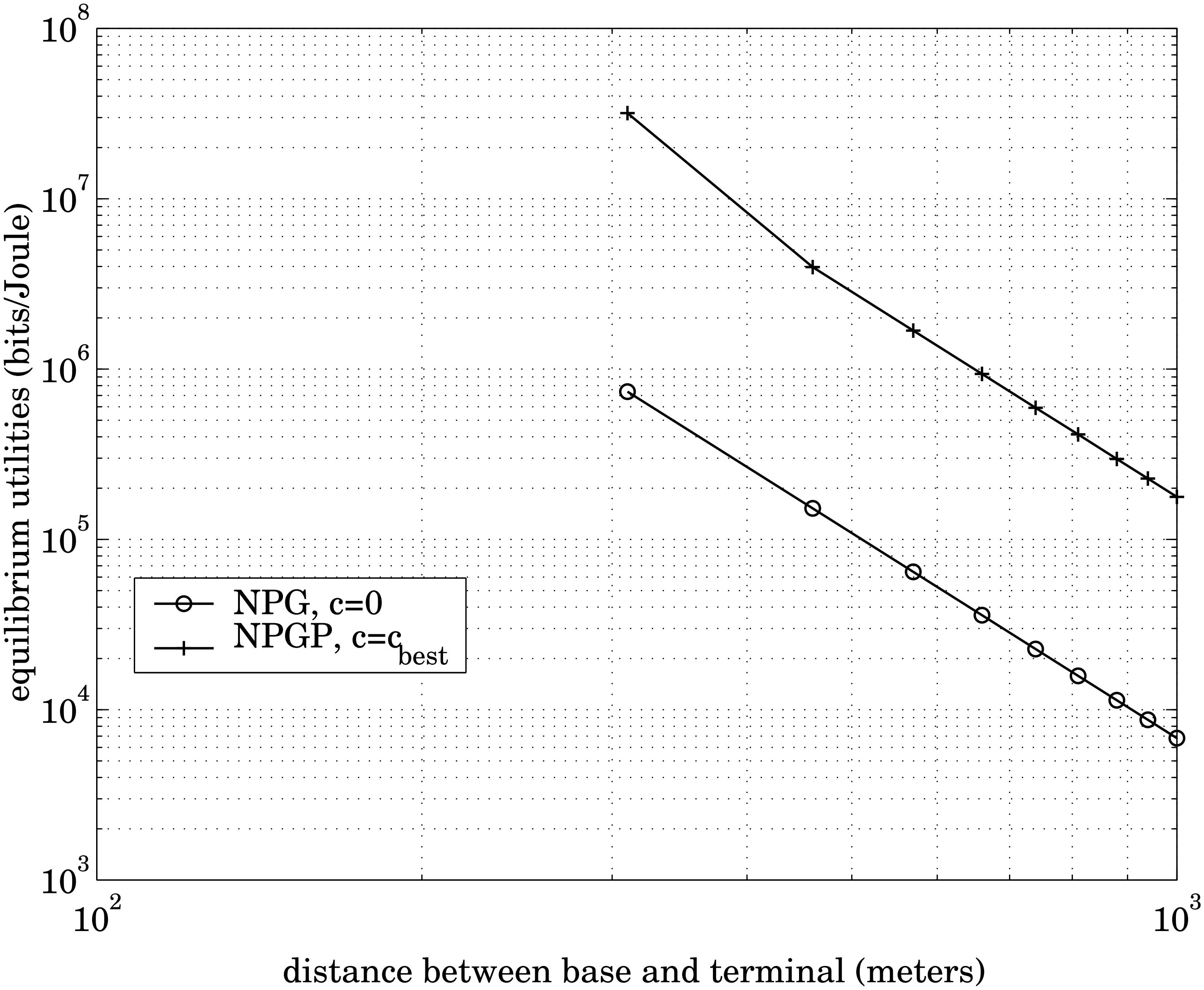

Figure 6: Utilities at equilibrium of NPG and NPGP with $c = c_{\text{best}}$.

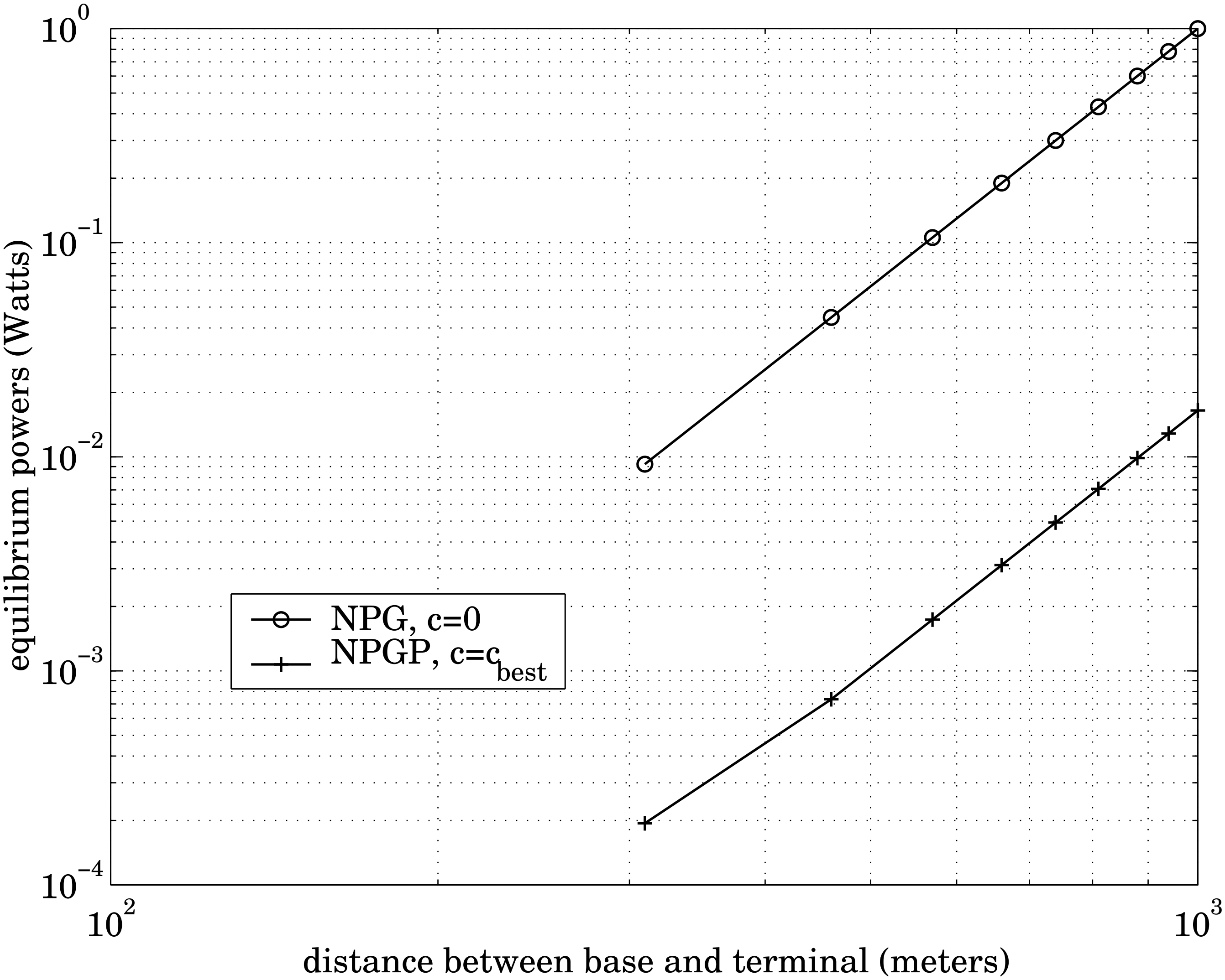

Figure 7: Powers at equilibrium of NPG and NPGP with $c = c_{\text{best}}$.