

## *Solving Linear Systems*

$$Ax = b$$

$$7x = 21$$

$$x = \frac{21}{7} = 3$$

$$x = 7^{-1} \times 21 = .142857 \times 21 = 2.99997$$

## *The MATLAB Backslash Operator*

$$AX = B$$

$$X = A \setminus B$$

$$XA = B$$

$$X = B / A$$

*A 3-by-3 Example*

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{aligned} 10x_1 - 7x_2 &= 7 \\ -3x_1 + 2x_2 + 6x_3 &= 4 \\ 5x_1 - x_2 + 5x_3 &= 6 \end{aligned}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & -0.1 & 6 \\ 0 & 2.5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6.1 \\ 2.5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2.5 \\ 6.1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2.5 \\ 6.2 \end{pmatrix}$$

$$6.2x_3 = 6.2$$

$$2.5x_2 + (5)(1) = 2.5.$$

$$10x_1 + (-7)(-1) = 7$$

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & -0.04 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$LU = PA$$

## *Permutation and Triangular Matrices*

*permutation matrix*

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Px = b$$

$$x = P^T b$$

*upper triangular matrix*

$$U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

*unit lower triangular matrix*

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 \\ 4 & 6 & 7 & 1 \end{pmatrix}$$

## *LU Factorization*

$$U = M_{n-1}P_{n-1} \cdots M_2P_2M_1P_1A$$

$$L_1L_2 \cdots L_{n-1}U = P_{n-1} \cdots P_2P_1A$$

$$L = L_1L_2 \cdots L_{n-1}$$

$$P = P_{n-1} \cdots P_2P_1$$

$$LU = PA$$

$$A = \begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.04 & 1 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.04 & 1 \end{pmatrix},$$

$$Ax = b$$

$$Ly = Pb$$

$$Ux = y$$

*Why Is Pivoting Necessary?*

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3.901 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6.001 \\ 2.5 \end{pmatrix}$$

$$(5 + (2.5 \cdot 10^3)(6))x_3 = (2.5 + (2.5 \cdot 10^3)(6.001))$$

$$(5 + 1.5000 \cdot 10^4)x_3 = (2.5 + 1.50025 \cdot 10^4)$$

$$1.5005 \cdot 10^4 x_3 = 1.5004 \cdot 10^4$$

$$x_3 = \frac{1.5004 \cdot 10^4}{1.5005 \cdot 10^4} = 0.99993$$

$$-0.001x_2 + (6)(0.99993) = 6.001$$

$$x_2 = \frac{1.5 \cdot 10^{-3}}{-1.0 \cdot 10^{-3}} = -1.5$$

$$10x_1 + (-7)(-1.5) = 7$$

$$x_1 = -0.35$$

## *Effect of Roundoff Errors*

*error*

$$e = x - x_*$$

*residual*

$$r = b - Ax_*$$

$$\begin{pmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.217 \\ 0.254 \end{pmatrix}$$

$$\frac{0.780}{0.913} = 0.854 \quad (\text{to three places})$$

$$\begin{pmatrix} 0.913 & 0.659 \\ 0 & 0.001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.254 \\ 0.001 \end{pmatrix}$$

$$x_2 = \frac{0.001}{0.001} = 1.00 \quad (\text{exactly}),$$

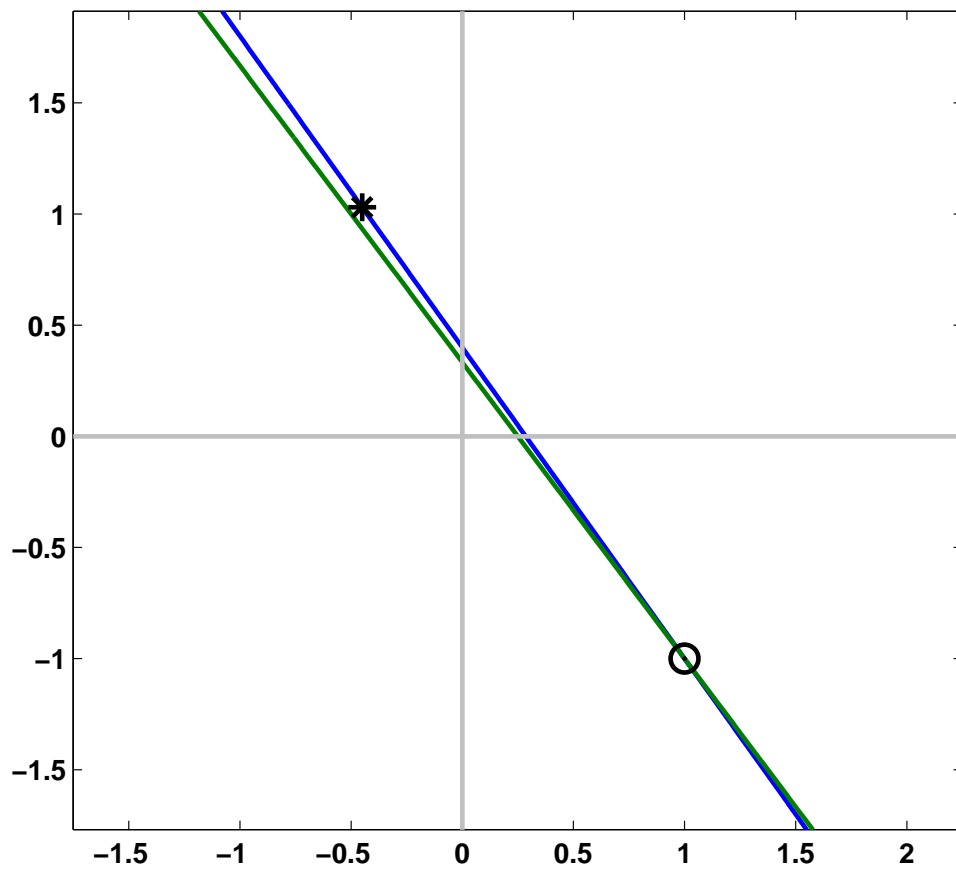
$$\begin{aligned} x_1 &= \frac{0.254 - 0.659x_2}{0.913} \\ &= -0.443 \quad (\text{to three places}). \end{aligned}$$

$$x_* = \begin{pmatrix} -0.443 \\ 1.000 \end{pmatrix}$$

$$\begin{aligned} r &= b - Ax_* = \begin{pmatrix} 0.217 - ((0.780)(-0.443) + (0.5 \\ 0.254 - ((0.913)(-0.443) + (0.6 \\ &= \begin{pmatrix} -0.000460 \\ -0.000541 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 0.913000 & 0.659000 \\ 0 & 0.000001 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.254000 \\ -0.000001 \end{pmatrix}$$

$$\begin{aligned} x_2 &= \frac{-0.000001}{0.000001} = -1.000000, \\ x_1 &= \frac{0.254 - 0.659x_2}{0.913} \\ &= 1.000000, \end{aligned}$$



*Gaussian elimination with partial pivoting is guaranteed to produce small residuals.*

## *Norms and Condition Numbers*

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

$$\|x\|_2 = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2}$$

$$\|x\|_\infty = \max_i |x_i|$$

$$\|x\| > 0 \text{ if } x \neq 0$$

$$\|0\| = 0$$

$$\|cx\| = |c|\|x\| \text{ for all scalars } c$$

$$\|x + y\| \leq \|x\| + \|y\|, \text{ (the } \textit{triangle inequality})$$

$$M = \max \frac{\|Ax\|}{\|x\|}$$
$$m = \min \frac{\|Ax\|}{\|x\|}$$

$$\kappa(A) = \frac{\max \frac{\|Ax\|}{\|x\|}}{\min \frac{\|Ax\|}{\|x\|}}$$

$$\kappa(A) = \|A\| \|A^{-1}\|$$

$$Ax = b$$

$$A(x + \delta x) = b + \delta b$$

$$\|b\| \leq M\|x\|$$

$$\|\delta b\| \geq m\|\delta x\|$$

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}$$

$$\kappa(A) \geq 1$$

$$\kappa(P) = 1$$

$$\kappa(cA) = \kappa(A)$$

$$\kappa(D) = \frac{\max |d_{ii}|}{\min |d_{ii}|}$$

$$A = \begin{pmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{pmatrix}$$

$$b = \begin{pmatrix} 4.1 \\ 9.7 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\|b\| = 13.8, \quad \|x\| = 1$$

$$\tilde{b} = \begin{pmatrix} 4.11 \\ 9.70 \end{pmatrix}$$

$$\tilde{x} = \begin{pmatrix} 0.34 \\ 0.97 \end{pmatrix}$$

$$\|\delta b\| = 0.01$$

$$\|\delta x\| = 1.63$$

$$\frac{\|\delta b\|}{\|b\|} = 0.0007246$$

$$\frac{\|\delta x\|}{\|x\|} = 1.63$$

$$\kappa(A) \geq \frac{1.63}{0.0007246} = 2249.4$$

$$\kappa(A) = 2249.4$$

$$\frac{\|b - Ax_*\|}{\|A\|\|x_*\|} \leq \rho\epsilon,$$

$$\frac{\|x - x_*\|}{\|x_*\|} \leq \rho\kappa(A)\epsilon$$

$$\|A\| = \max \frac{\|Ax\|}{\|x\|}$$

$$\|A\|_1 = \max_j \sum_i |a_{i,j}|$$

$$\|A\|_\infty = \max_i \sum_j |a_{i,j}|$$

$$(A + E)x_* = b$$

$$\frac{\|E\|}{\|A\|} = \rho\epsilon$$

$$b - Ax_* = Ex_*$$

$$\|b - Ax_*\| = \|Ex_*\| \leq \|E\|\|x_*\|$$

$$\frac{\|b - Ax_*\|}{\|A\|\|x_*\|} \leq \rho\epsilon$$

$$x - x_* = A^{-1}(b - Ax_*)$$

$$\|x - x_*\| \leq \|A^{-1}\| \|E\| \|x_*\|$$

$$\frac{\|x - x_*\|}{\|x_*\|} \leq \rho \|A\| \|A^{-1}\| \epsilon$$

$$\frac{\|x - x_*\|}{\|x_*\|} \leq \rho \kappa(A) \epsilon$$

## *Sparse Matrices and Band Matrices*

$$\begin{pmatrix} b_1 & c_1 & & & & & \\ a_1 & b_2 & c_2 & & & & \\ & a_2 & b_3 & c_3 & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & a_{n-2} & b_{n-1} & c_{n-1} & \\ & & & & a_{n-1} & b_n & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$
$$= \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix}$$

## *PageRank and Markov Chains*

$$g_{ij} = 1, \text{ if node}_j \rightarrow \text{node}_i$$

$$c_j = \sum_i g_{ij}, \quad r_i = \sum_j g_{ij}$$

$$a_{ij} = pg_{ij}/c_j + \delta, \text{ where } \delta = (1 - p)/n.$$

$$x = Ax$$

$$\sum_i x_i = 1$$

$$(I - A)x = 0$$

$$x = Ax$$

$$x = (p\tilde{G} + \delta ee^T) x$$

$$e^T x = 1$$

$$(I - p\tilde{G})x = \delta e$$

```
[U,G] = surfer('http: ... ');  
c = sum(G)  
D = spdiags(1./c',0,n,n)  
p = .85  
delta = (1-p)/n  
e = ones(n,1)  
I = speye(n,n)  
x = (I - p*G*D)\(delta*e)
```

## *Power method*

```
while termination_test
    x = A*x;
end
```

## *Inverse iteration*

$$x = (I - A) \setminus e$$

$$x = x / \text{sum}(x)$$