Homework 3

CS 290 H : Graph Laplacians and Spectra : Spring 2014

Due by Wednesday, May 21

Turn in all your homework on paper, either in class or in the homework box in the Computer Science Department office. For Matlab exercises and programs, turn in (1) listings of any code you wrote; (2) any plots and relevant output; (3) a "diary" transcript of the relevant part of a Matlab session.

Problem 1. Pseudoinverse. Recall the definition of the pseudoinverse of a symmetric matrix A (see the Index):

$$A^+ = \sum_{\lambda_i \neq 0} \frac{1}{\lambda_i} w_i w_i^T,$$

where the sum is taken over the nonzero eigenvalues of A. Prove that if x is orthogonal to the null space of A (i.e. $x^T w_i = 0$ whenever $\lambda_i = 0$), then

$$A^+ A x = A A^+ x = x.$$

Problem 2. Semidefinite ordering. Recall from the Index the definition of the semidefinite ordering relationship $A \succeq B$ between two symmetric matrices A and B, where $A \succeq 0$ means A is positive semidefinite.

2(a) Prove that if $A \succeq 0$ then $A^+ \succeq 0$.

2(b) Prove that if $A \succeq B \succeq 0$ and $\mathrm{null}(A) = \mathrm{null}(B) = \mathrm{span}(\mathbf{1})$, then $A^+ \preceq B^+$. (Hint: Take $x^T\mathbf{1} = 0$, let $y = A^+x - B^+x$, and look at $y^T B y$.)

Problem 3. Conjugate gradient algorithm. This is another open-ended problem whose goal is for you to get some intuition about the behavior of the conjugate gradient algorithm for graph Laplacians. You are to experiment with CG in Matlab. You may use Matlab's built-in `pcg()` routine or write your own (simpler) CG in a dozen lines of Matlab. In the latter case you have the advantage that you can include reorthogonalization against the constant vector w_1 if necessary (though I guess you could add that to Matlab's `pcg.m` source too). There are two experiments to run:

3(a) Experiment with the Laplacian matrices of the same sample graphs you used for the Lanczos experiments in Homework 2, plus others if you want. Try CG three ways: with no preconditioning, with Jacobi preconditioning (that is, the preconditioner is just the diagonal of the matrix), and with incomplete Cholesky preconditioning from Matlab's `R = cholinc(L,'0')`. Make and turn in semilog plots of convergence history of the relative residual as I did in class on Wednesday, May 7. Also, for each Laplacian, what is the finite condition number $\kappa_f(L)$, that is, the ratio of the largest to smallest nonzero eigenvalue? Your Lanczos experiments from last time give you the data to answer this.

3(b) Generate two sequences of Laplacian matrices of graphs of increasing sizes as follows: The first sequence is the "model problem" square grid graph; Matlab's `L = laplacian(grid5(k));` gives you the k-by-k grid with $n = k^2$ vertices. The second sequence is flat random graphs, generated with Matlab's `sprandsym()`. Set the parameters so that the second sequence has graphs of about the same sizes and numbers of edges as the first sequence. *Important:* For the second sequence, use just the largest connected component of the graph you generate; see Matlab's `components()` routine.

For both sequences, convert the matrices to unweighted Laplacians (with `laplacian()`). Experiment with CG, both unpreconditioned and preconditioned. Your goal is to get enough data to estimate for each sequence the asymptotic scaling of the number of CG iterations needed to reach some fixed residual reduction (say 10^{-6}) as a function of n, the dimension of the matrix. Is the scaling different for the two sequences? Is it different for preconditioned and unpreconditioned CG?

Write a paragraph or so analyzing your results and stating your conclusions.