Faraday Waves

Model
A model for Faraday waves due to Ron Lifshitz is

\[ \frac{\delta u}{\delta t} = \epsilon u - c(\Delta + I)²(\Delta + q²I)²u + \alpha u² - u³ \] (1)

Finite Difference Scheme
A finite difference scheme for equation (1) is

\[ \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} = \epsilon u_{ij}^n - c(\Delta + I)²(\Delta + q²I)²u_{ij}^{n+1} + \alpha(u_{ij}^n)² - (u_{ij}^n)³ \]
\[ c[I + \Delta t(\Delta + I)²(\Delta + q²I)²]u_{ij}^{n+1} = u_{ij}^n + \Delta t[\epsilon u_{ij}^n + \alpha(u_{ij}^n)² - (u_{ij}^n)³] \] (2)

Equation (2) can be rewritten as a matrix equation

\[ (I + c\Delta tA)u = f \] (3)

where

\[ A = (\Delta + I)²(\Delta + q²I)² \] (4)

and

\[ f = u_{ij}^n + \Delta t[\epsilon u_{ij}^n + \alpha(u_{ij}^n)² - (u_{ij}^n)³] \]

Assume \( \{v_i\}_{i=1,n} \) are the eigenvectors of \( A \), and they form a basis

\[ Av_i = \lambda_i v_i \]

then

\[ (I + \Delta tA)v = v + \Delta tAv \]
\[ \dot{v} + \Delta t \lambda v = (1 + \lambda \Delta t) v \]

so equation (3) can be solved using discrete Fourier transforms on both \( u \) and \( f \) to obtain

\[ \hat{u}_{mn} = \frac{\hat{f}_{mn}}{(1 + \Delta t \lambda)} \]

then using the discrete inverse Fourier transform to obtain \( u \).

**Fourier Transform Solution**

The discrete inverse Fourier transform in \( x \) and \( y \) is

\[ u_{jl} = \frac{1}{JL} \sum_{m=0}^{J-1} \sum_{n=0}^{L-1} \hat{u}_{mn} e^{-\frac{2\pi im}{J} x} e^{-\frac{2\pi in}{L} y} \]

(5)

Similarly,

\[ f_{jl} = \frac{1}{JL} \sum_{m=0}^{J-1} \sum_{n=0}^{L-1} \hat{f}_{mn} e^{-\frac{2\pi im}{J} x} e^{-\frac{2\pi in}{L} y} \]

(6)

The eigenvalues of the Laplacian are

\[ \lambda_L = (-\theta_x^2 - \theta_y^2) * \hat{u} \]

(7)

where \( \theta_x = \frac{2\pi m}{J} \) and \( \theta_y = \frac{2\pi n}{L} \)

so the eigenvalue of the linear operator \( A \) in equation (4) are

\[ \lambda = (-\theta_x^2 - \theta_y^2 + 1)^2 * (-\theta_x^2 - \theta_y^2 + Q^2)^2 * \hat{u} \]

(8)

**Stability Condition**

The finite difference scheme is stable if the amplification factor

\[ |g(\theta, \Delta t, \Delta x)| = \frac{1 + \Delta t(\epsilon + \alpha - 1)}{1 + \Delta t \lambda} \leq 1 + K \Delta t \]

where \( K \) is a constant. The amplification factor \( g \) is less than one if

\[ (\epsilon + \alpha - 1) < \lambda \]
Accuracy

Taylor series expansion of the $\Delta$ operator gives

$$\Delta = \frac{v^n_m + \Delta x v_x + \frac{\Delta x^2}{2} v_{xx} - 2v^n_m + v^n_m - \Delta x v_x + \frac{\Delta x^2}{2} v_{xx} + O(\Delta x^4)}{\Delta x^2} + \frac{v^n_m + \Delta y v_y + \frac{\Delta y^2}{2} v_{yy} - 2v^n_m + v^n_m - \Delta y v_y + \frac{\Delta y^2}{2} v_{yy} + O(\Delta y^4)}{\Delta y^2} = v_{xx} + \frac{1}{3} \Delta x^2 v_{xxxx} + v_{yy} + \frac{1}{3} \Delta y^2 v_{yyyy} + O(\Delta x^4, \Delta y^4)$$ (9)

Taylor series expansion of the finite difference approximation of the time derivate gives

$$\frac{\delta u}{\delta t} = \frac{v^n_m + \Delta t v_t + \frac{\Delta t^2}{2} v_{tt} - v^n_m + O(\Delta t^3)}{\Delta t} = v_t + \frac{\Delta t}{2} v_{tt} + O(\Delta t^2)$$ (10)

The order of accuracy of this finite difference method is $O(\Delta t, \Delta x^2, \Delta y^2)$. 