

Review of Last Lecture

- Recursive definitions
 - Recursively defined functions
 - Recursively defined sets
- Basic counting
 - Production rules: $n_1 \cdot n_2 \cdot \dots \cdot n_m$ ways
 - Sum rules: $n_1 + n_2 + \dots + n_m$ ways
 - Inclusive-exclusive principles:
 - $|A \cup B| = |A| + |B| - |A \cap B|$
 - $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$
- Pigeonhole Principles

*If N **objects** are placed into k **boxes**, then there is **at least one box containing at least** $\lceil N/k \rceil$*

§ 4.3 Permutations and Combinations

- Ex1: $S = \{1, 2, 3\}$, how many strings of length 3 without repeated characters can be formed from S ?
 - **A**: $3 \cdot 2 \cdot 1 = 6$ (product rule)
- DEF1: A *permutation* of a set is an **ordered** assignment of its elements.
- DEF2: A *r-permutation* of a set is an **ordered** assignment of r elements of a set
- Theorem1: The number of r -permutation of a set with n distinct elements is
$$\mathbf{P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1) = n! / (n-r)!}$$
 - How to prove it? (Hint: by product rule)
 - **$P(n, n) = n!$**

Permutations and Combinations

- Ex2: How many permutations of the letters {A, B, C, D, E, F, G, H} contain the string "ABC"?
 - Hint: treat "ABC" as one "letter", then how many permutations of {"ABC", D, E, F, G, H}?
 - A: $6! = 720$
 - How about "A, B, C" must be adjacent to each other?
A: $6! \cdot 3! = 4320$
- Note: What if the order doesn't matter?
- Ex3: 10 players, how many ways to pick 5 of them?
- DEF3: An *r-combination* of a set is an **unordered** selection of r elements from the set.

Permutations and Combinations

- Theorem2: The number of r-combinations of a set with n elements is

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

- How to prove it?
- proof: The r-permutations of a set can be constructed by two steps:

(1) form all r-combinations --- C(n, r) ways

(2) Ordering elements in each r-combination --- r! ways

$$\Rightarrow P(n, r) = C(n, r) \cdot r!$$

$$\Rightarrow C(n, r) = P(n, r) / r! = \frac{n!}{(n-r)! \cdot r!}$$

Permutations and Combinations

- Theorem3: $C(n,r) = C(n, n-r)$

- How to prove it?

- proof: $C(n, r) = \frac{n!}{(n-r)! \cdot r!}$ ← by Theorem2

$$C(n, n-r) = \frac{n!}{(n - (n-r))! \cdot (n-r)!} = \frac{n!}{(n-r)! \cdot r!}$$

$$\therefore C(n, r) = C(n, n-r)$$

- Ex4: 10 people, ways to pick 6 of them are as many as ways to pick 4 of them!

$$C(10, 6) = ? \quad C(10, 4) = ?$$

§ 5.1 An Introduction to Discrete Probability

- What is the probability that a person picks the correct 6 numbers out of 40 to win a supper lottery price?
- DEF1:
 - *Experiment*: a procedure that yields one of a given set of possible outcomes
 - *Sample space S*: the set of possible outcomes
 - *Event E*: a subset of the sample space, $E \subseteq S$
 - *Probability of E*: if S is finite, then the probability of E is

$$P(E) = \frac{|E|}{|S|}$$

- Note: this definition is based on the **assumption** that all outcomes are equally likely.

Discrete Probability





- Ex1: Toss a coin twice, what's the probability of getting two heads?
 - Sample space $S = \{HH, HT, TH, TT\}$
 - Event $E = \{HH\}$
 - $P(E) = |E| / |S| = 1/4$
 - If $E = \{ \text{at least one tail} \}$, what is $P(E)$?
- Note
 - $0 \leq P(E) \leq 1$ since $E \subseteq S$
 - $P(\emptyset) = 0$ since $|\emptyset| = 0$
 - $P(S) = 1$ since $|S| / |S| = 1$

Discrete Probability

- Ex2: When two dice are rolled, what's the probability that the sum of the numbers on the two dice is 7?
 - step 1: how many possible outcomes totally? (product rule)
 $\Rightarrow |S| = 6 \cdot 6 = 36$
 - step 2: How many possible outcomes that the sum of two dice is 7? (list them all) $\Rightarrow |E| = 6$
 - step 3: $P(E) = |E| / |S| = 6/36 = 1/6$
- Ex3: What is the probability that a person picks the correct 6 numbers out of 40 to win a supper lottery price?
 - $|S| = C(40, 6) = 40! / (34! 6!) = 3,838,380$
 - $|E| = 1$
 - $P(E) = |E|/|S| = 1/ 3,838,380 \approx 0.00000026!$



Discrete Probability

- A standard deck of cards
 - 52 cards
 - 4 suits (H-  D-  C-  S- )
 - 13 kinds for each suit: 2~10, J, Q, K, A
- Ex4: What is the probability that a 5 card poker hand contains 4 of a kind?
 - $|S| = C(52, 5) = 2,598,960$, $|E| = C(13, 1) \cdot C(48, 1) = 624$
 - $P(E) = |E| / |S| = 624 / 2,598,960 \approx 0.00024$
- Ex5: What is the probability that a poker hand contains a full house (3 one kind, 2 another kind)?
 - $|E| = P(13, 2) \cdot C(4, 3) \cdot C(4, 2) = 3744$
 - $P(E) = |E| / |S| = 3744 / 2,598,960 \approx 0.0014$

Discrete Probability

- Ex6: 50 balls have number 1~50 on them. What's the probability that the numbers 11, 4, 17, 39, 23 are drawn in that order (1) If the ball selected is not returned to the bin (2) If the ball selected is returned to the bin?
 - Note: The difference between "return" and "not return", i.e. allow repeat or not.
 - (1) $|S| = 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254,251,200$, $|E| = 1$
 $\Rightarrow P(E) = |E| / |S| = 1/254,251,200 \approx 0.0000000039$
 - (2) $|S| = 50^5 = 312,500,000$, $|E| = 1$
 $\Rightarrow P(E) = |E| / |S| = 1/312,500,000$

Combination of Events

- Theorem1 (how to prove them?)

(1) $P(\bar{E}) = 1 - P(E)$

(2) $E_1 \subseteq S, E_2 \subseteq S$, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

- proof:

(1)
$$P(\bar{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - P(E)$$

(2)
$$\begin{aligned} P(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} \\ &= \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \end{aligned}$$

Combination of Events

- Ex7: What's the probability that a 5 card poker hand contains at least one K?
 - Hint: Let $E = \{\text{no K's}\} \Rightarrow \overline{E} = \{\text{at least one K}\}$
 - **A**:
$$P(\overline{E}) = 1 - P(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{C(48, 5)}{C(52, 5)}$$
- Ex8: What the probability that a 5 card poker hand contains a black K?
 - Hint: Let $E_1 = \{\text{contains a spade K}\}$, $E_2 = \{\text{contains a club K}\}$
 - **A**:
$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$
$$= \frac{C(51, 4)}{C(52, 5)} + \frac{C(51, 4)}{C(52, 5)} - \frac{C(50, 3)}{C(52, 5)}$$

§ 6.1 Recurrence Relations

- Ex1 Recall the Fibonacci sequence: 1,1,2,3,5,8,13,...

Two ways to define it:

- Recursive definition:

$$(B) f_0 = 1, f_1 = 1$$

$$(R) f_n = f_{n-1} + f_{n-2} \quad \text{if } n > 2$$

- Directly use generating function:

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

- Which way is easier to come up with?

Recurrence Relations

- DEF1:

- **Recurrence relation** is an equation that express the *n*th term of a sequence as a function of the previous terms:

$$x_n = f(x_{n-1}, x_{n-2}, \dots, x_0)$$

- **Initial conditions** specify the terms that precede the first term where the recurrence relation take effect
- A **solution** of a recurrence relation is a sequence $\{a_n\}$ that satisfies the recurrent relation

- Ex1 cont.:

- Recurrent relation: $f_n = f_{n-1} + f_{n-2}$ if $n > 2$

- Initial conditions: $f_0 = 1, f_1 = 1$

- solution of this recurrent relation: $\{f_n\}$, where

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

Recurrence Relations

- How to compute a solution for a recurrence relation given initial conditions?
- Naive way:
 - step1: enumerate couples of terms
 - step2: spot a pattern and guess the answer
 - step3: prove the answer (by induction)
 - Ex2: $a_0 = 0, a_n = a_{n-1} + 2n - 1$
 - step1: $a_0 = 0, a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, \dots$
 - step2: looks like $a_n = n^2$
 - step3: prove that $P(n) = "a_n = n^2"$ by induction

Recurrence Relations

- Ex2 cont:

- proof: Let $P(n) = "a_n = n^2"$

I. Base: $n = 0, a_0 = 0, 0^2 = 0, \therefore P(0)$ is true

II. I.H.: assume $P(k)$ is true, i.e. $a_k = k^2$

III. I. S.: when $n = k+1,$

$$a_{k+1} = a_k + 2(k+1) - 1 \quad // \text{ by the definition of } a_n$$

$$= k^2 + 2k + 1 \quad // \text{ by I.H}$$

$$= (k+1)^2 \quad \therefore P(k+1) \text{ is true}$$

\therefore Based on PMI, $a_n = n^2$

- Note: it is not easy to guess a pattern in most cases, we will discuss a more general ways to compute the solution for a recurrence relation in §6.2

Model Problems with Recurrence Relations

- Ex3: compound interest problem

Initial deposit: \$10,000 Annual interest: 11%

How much will be in the account after 30 years?

- $P_0 = 10,000$
 - $P_n = P_{n-1} + 0.11 \cdot P_{n-1} = 1.11 \cdot P_{n-1} = \dots = (1.11)^n P_0$
 - $P_{30} = (1.11)^{30} \cdot 10,000 = \$228,922,97$
- Ex4: Rabbits problem (P_{403} /Example 4)
rules: can't breed until 2 months old, after that, each pair produces another pair every month.
- Let f_n be the number of pairs of rabbits after n months
 - $f_1 = 1, \quad f_2 = 1$
 - $f_n = f_{n-1} + f_{n-2}$

Recurrent Relations

- Ex5: The tower of Hanoi Problem: (P₄₀₄/Example 5)
 - Task: Get all disks from peg 1 to peg 2.
 - Rules: (a) Only move 1 disk at a time.
(b) Never set a larger disk on a smaller one.
 - Hint: if we can solve the problem with size $n-1$, how to solve the problem with size n ?
 - solution: Let H_n be the number of moves for a stack of n disks.
 - (1) Move top $n-1$ disks to peg 3. (H_{n-1} moves)
 - (2) Move bottom disk to peg 2. (1 move)
 - (3) Move top $n-1$ to peg 2. (H_{n-1} moves)
$$\therefore H_n = 2H_{n-1} + 1$$
 - Try the case that $n=3$

Model Problems with Recurrence Relations

- Ex6: Model the number of bit strings of length n that do not have two consecutive 0's.

- Let a_n be the number of such bit strings
- $a_1 = 2$ ("0", "1")
- $a_2 = 3$ ("01", "10", "11")
- $a_n = ?$

➤ case1: end with 1, $a_n = a_{n-1}$



➤ case2: end with 0

- case 2.1: end with 00 $a_n = 0$



- case 2.2: end with 10 $a_n = a_{n-2}$



$\therefore a_n = a_{n-1} + a_{n-2}$ (the sum rule)