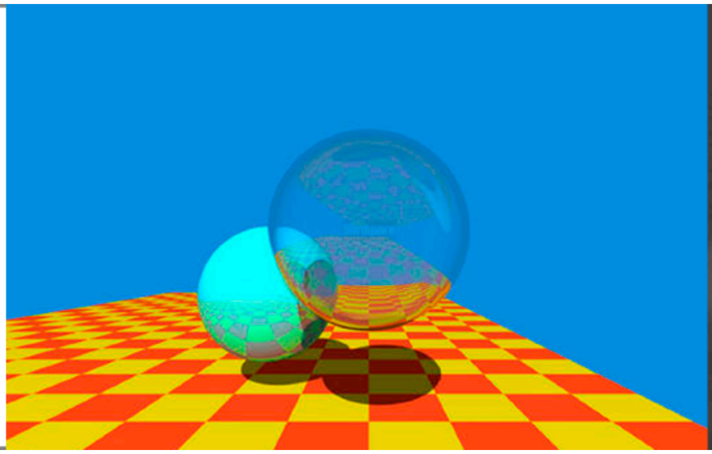
Introduction to Computer Graphics

GAMES101, Lingqi Yan, UC Santa Barbara

Lecture 3: Transformation






http://www.cs.ucsb.edu/~lingqi/teaching/games101.html

Announcements

- Assignment 0 will be released soon
- Do not register the homework submission system with QQ email

Last Lecture

- Vectors
  - Basic operations: addition, multiplication
- Dot Product
  - Forward / backward (dot product positive / negative)
- Cross Product
  - Left / right (cross product outward / inward)
- Matrices

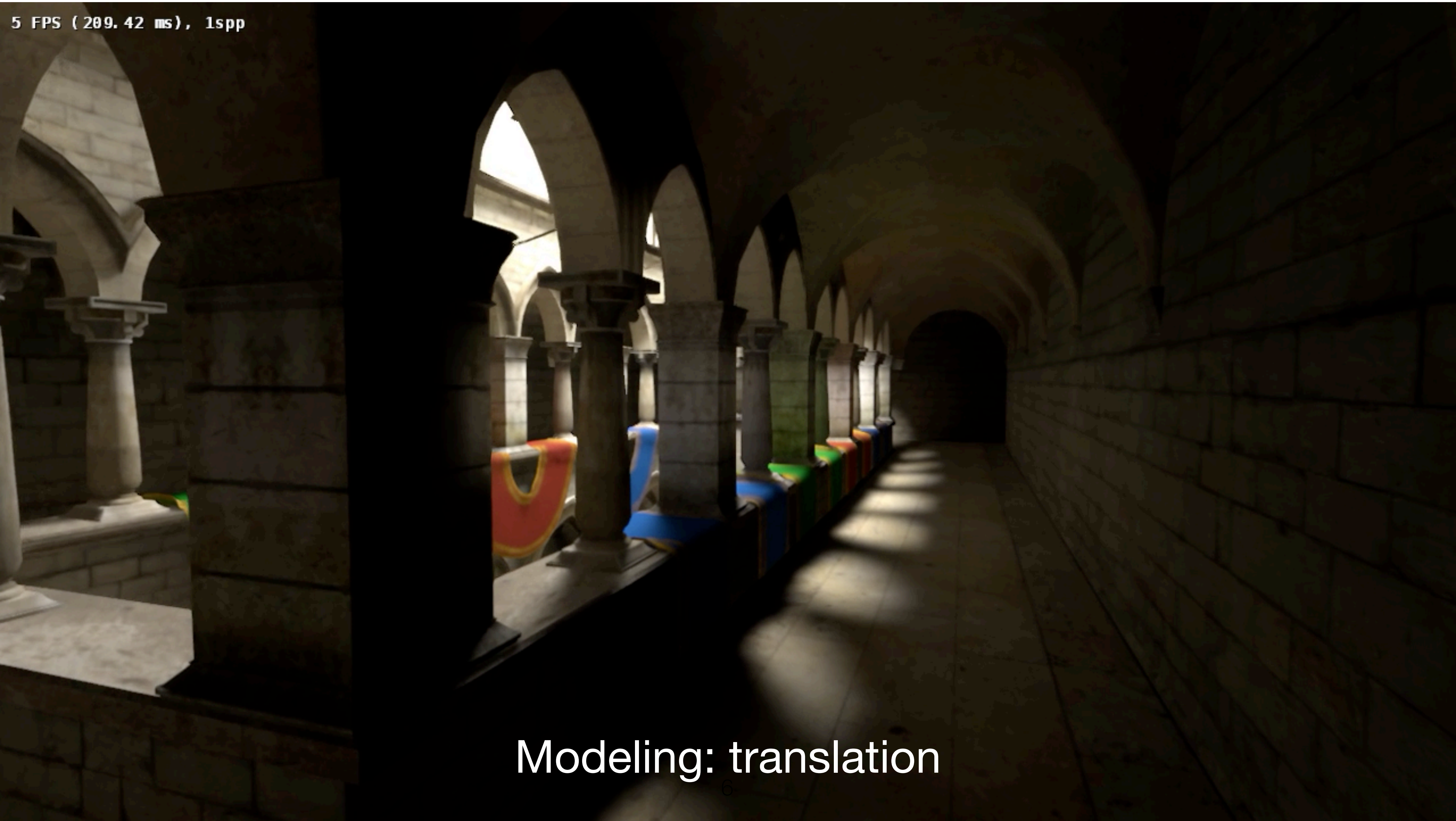
This Week

- Transformation!
- Today
  - Why study transformation
  - 2D transformations: rotation, scale, shear
  - Homogeneous coordinates
  - Composing transforms
  - 3D transformations

Today

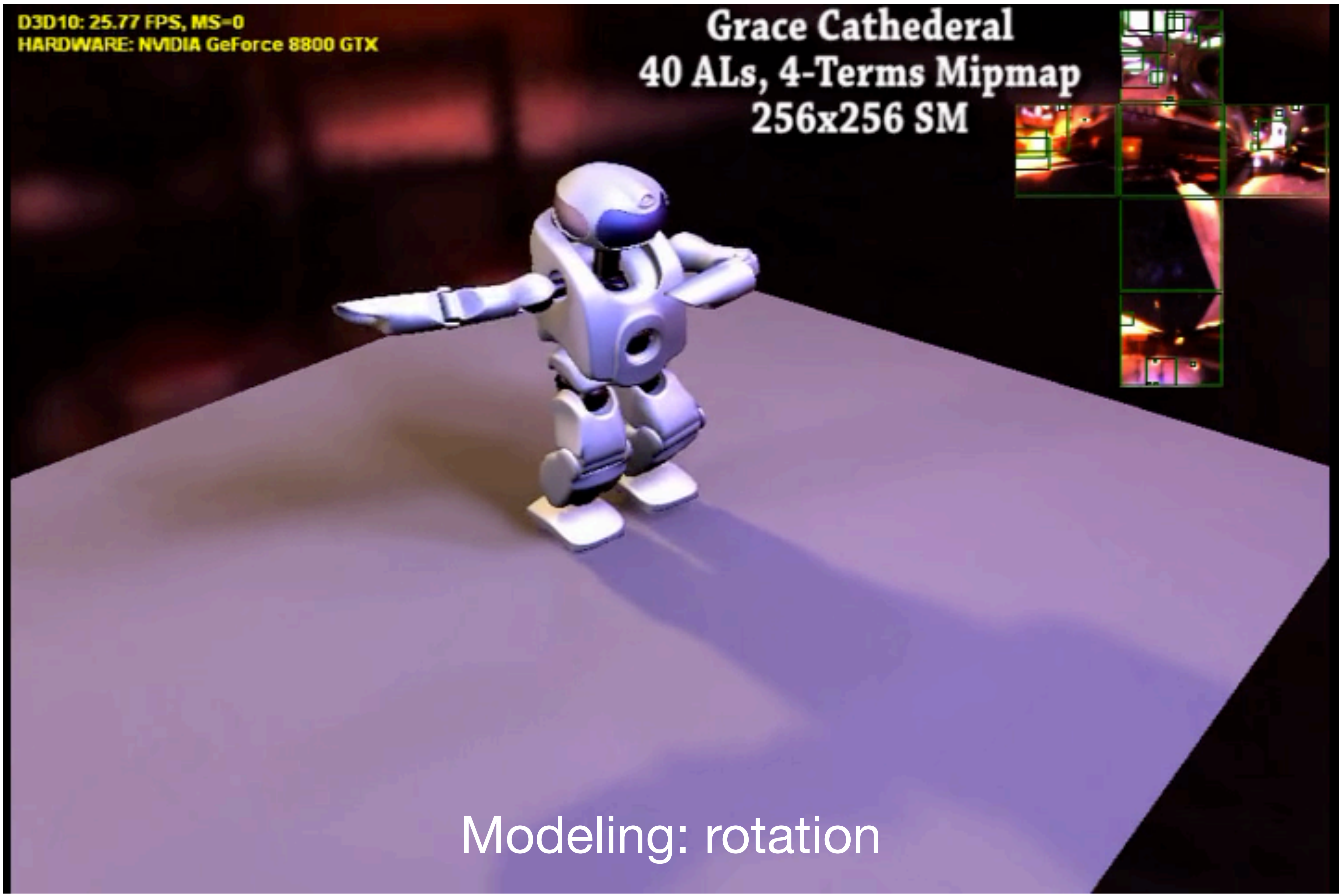
- Why study transformation
  - Modeling
  - Viewing
- 2D transformations
- Homogeneous coordinates

Why Transformation?



Modeling: translation

Why Transformation?

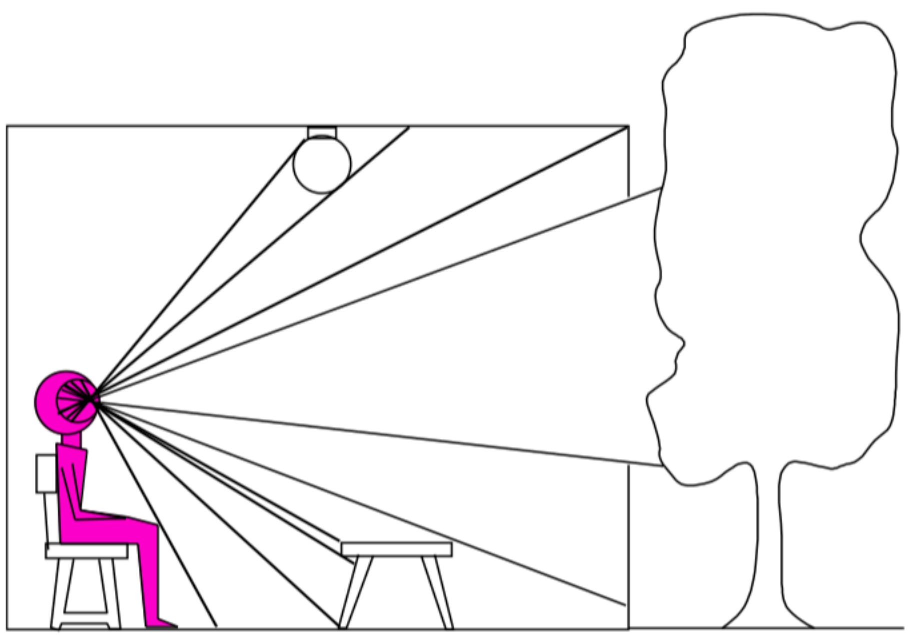


Why Transformation?

Modeling: scaling

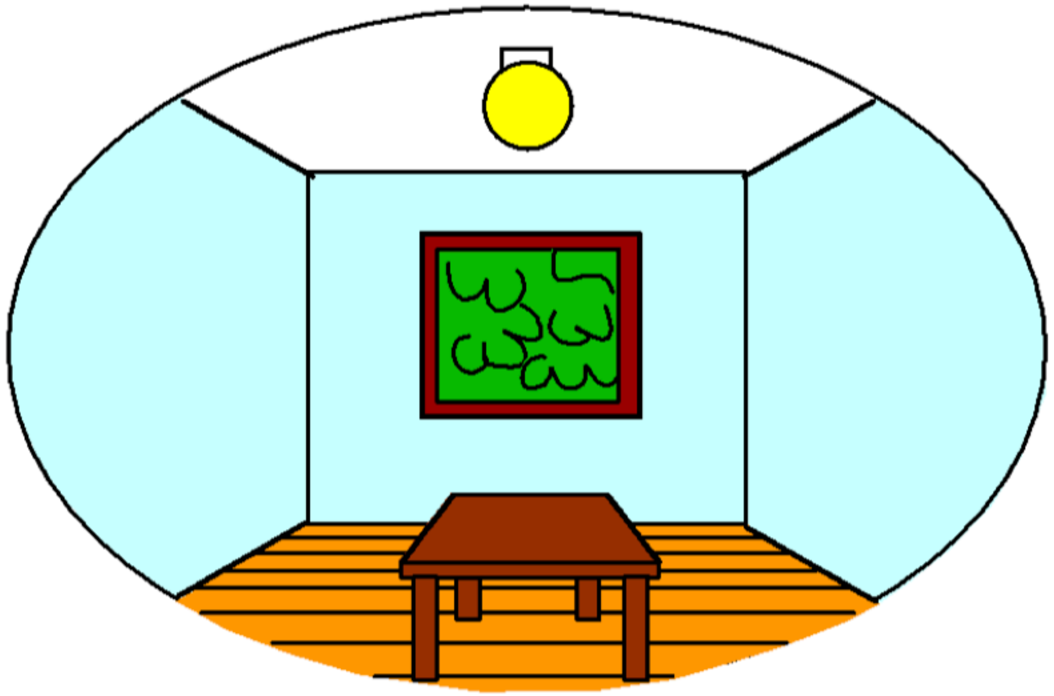
Why Transformation?

3D world



Point of observation

2D image



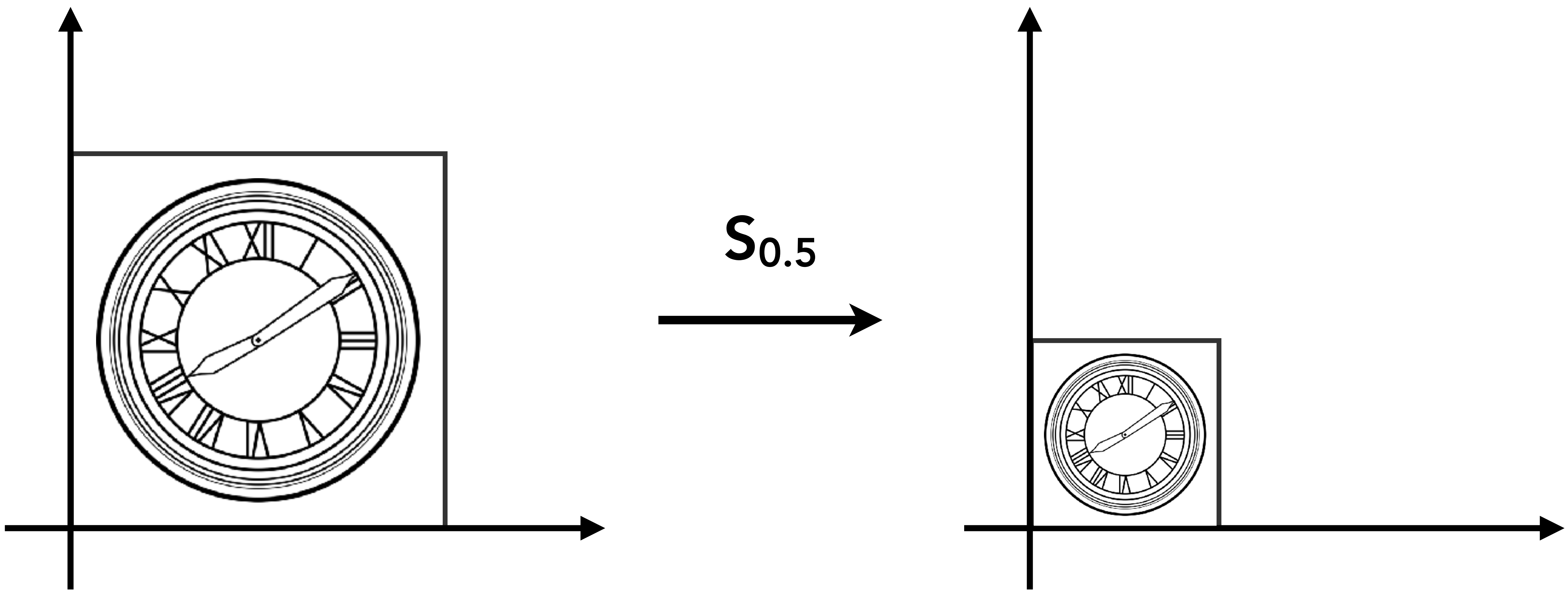
Figures © Stephen E. Palmer, 2002

Viewing: (3D to 2D) projection

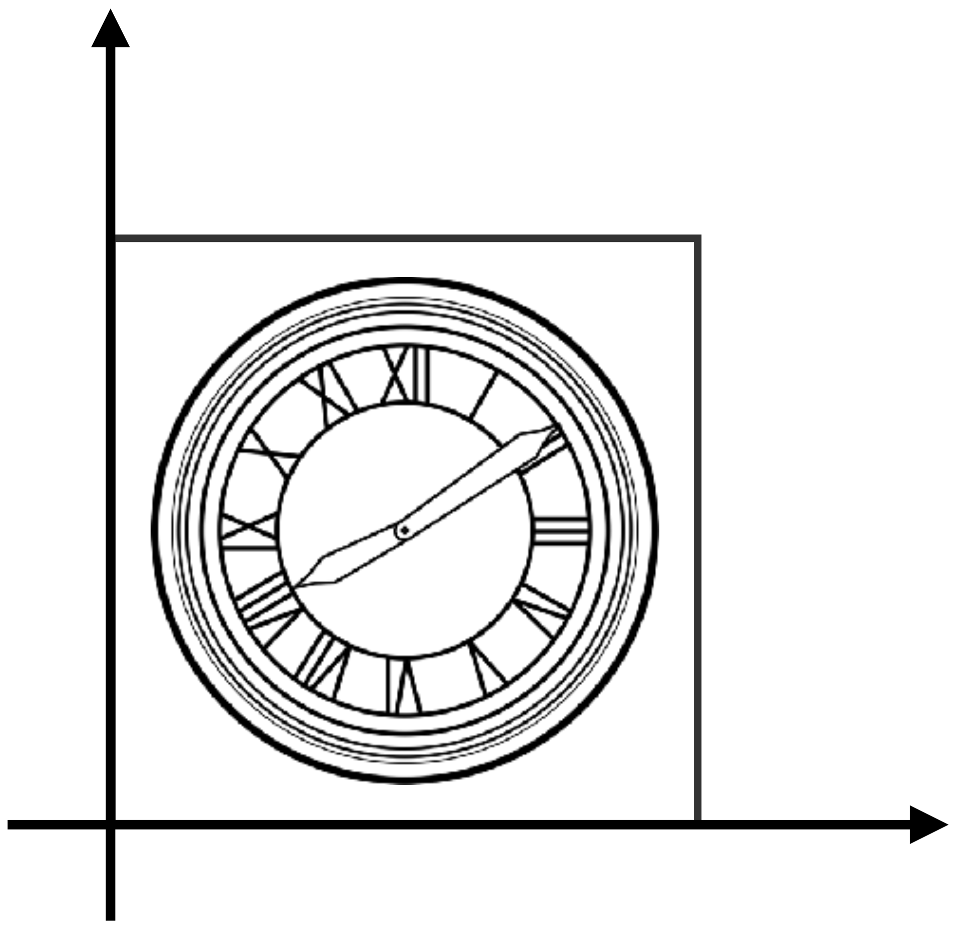
Today

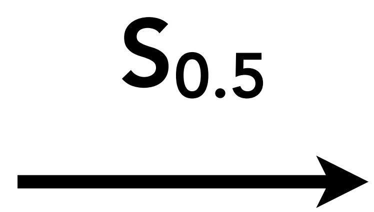
- Why study transformation
- 2D transformations
  - Representing transformations using matrices
  - Rotation, scale, shear
- Homogeneous coordinates

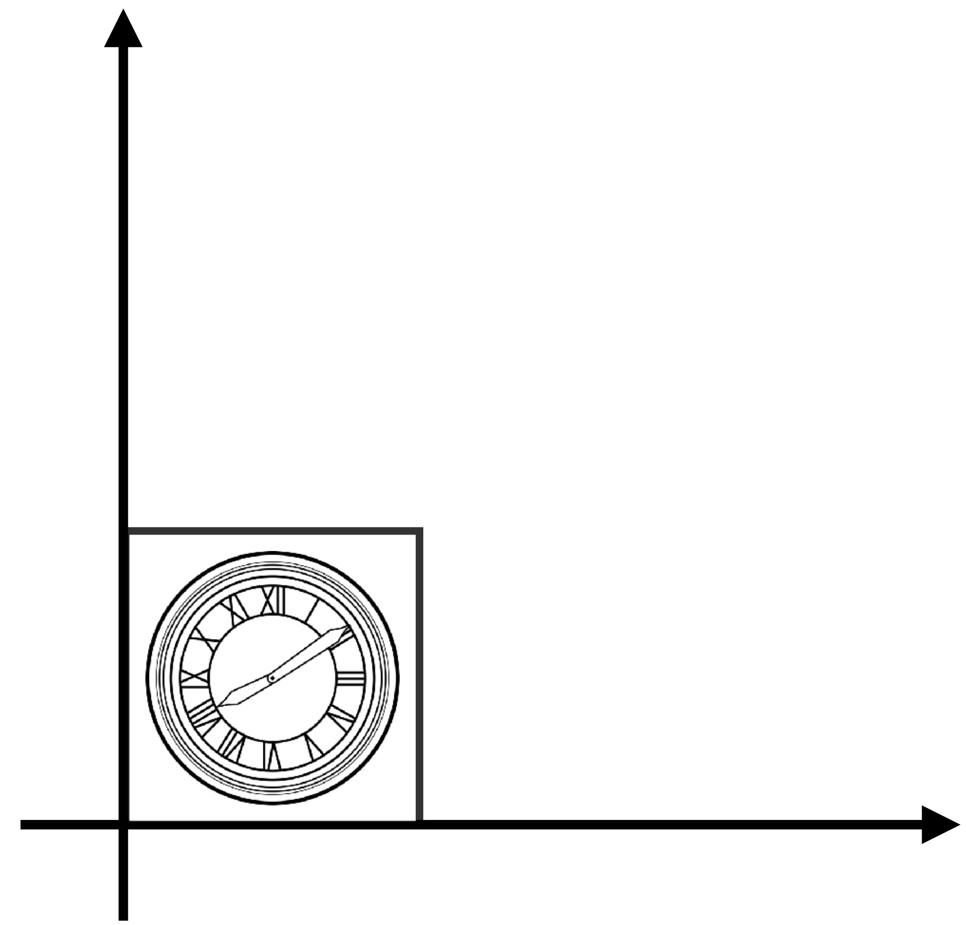
Scale



Scale Transform



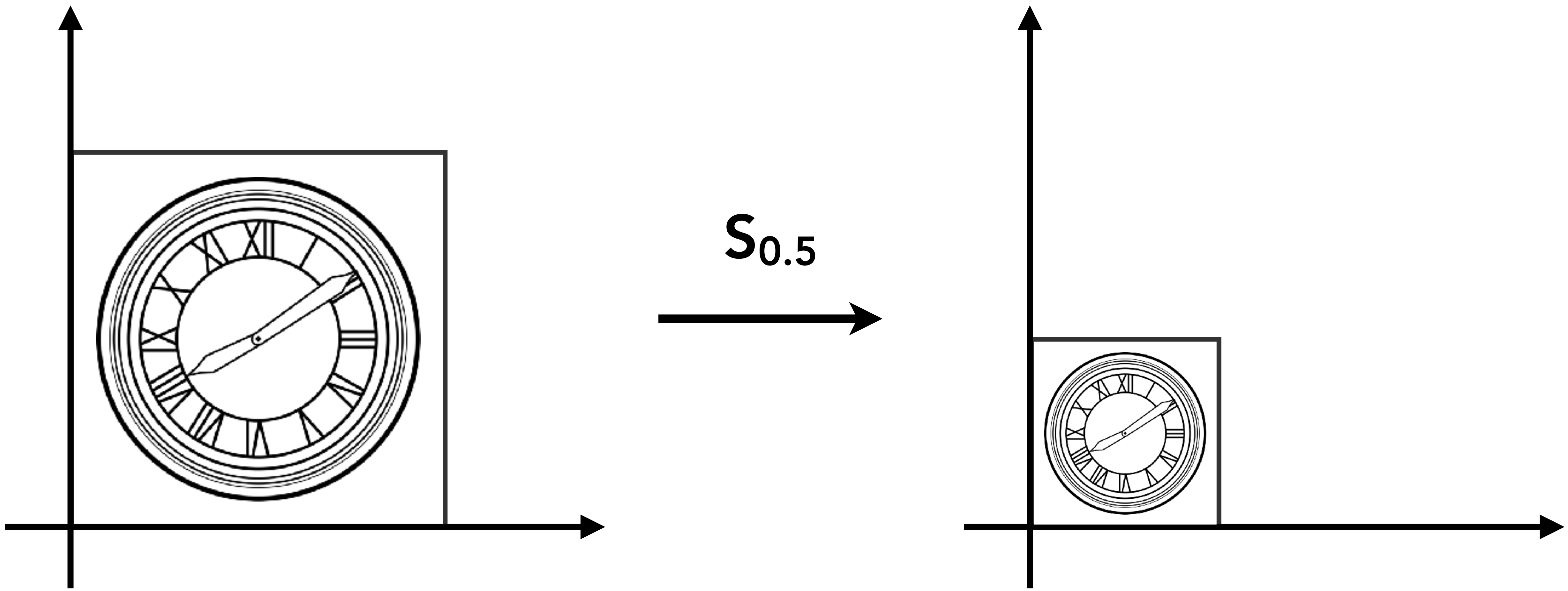




$$x' = sx$$

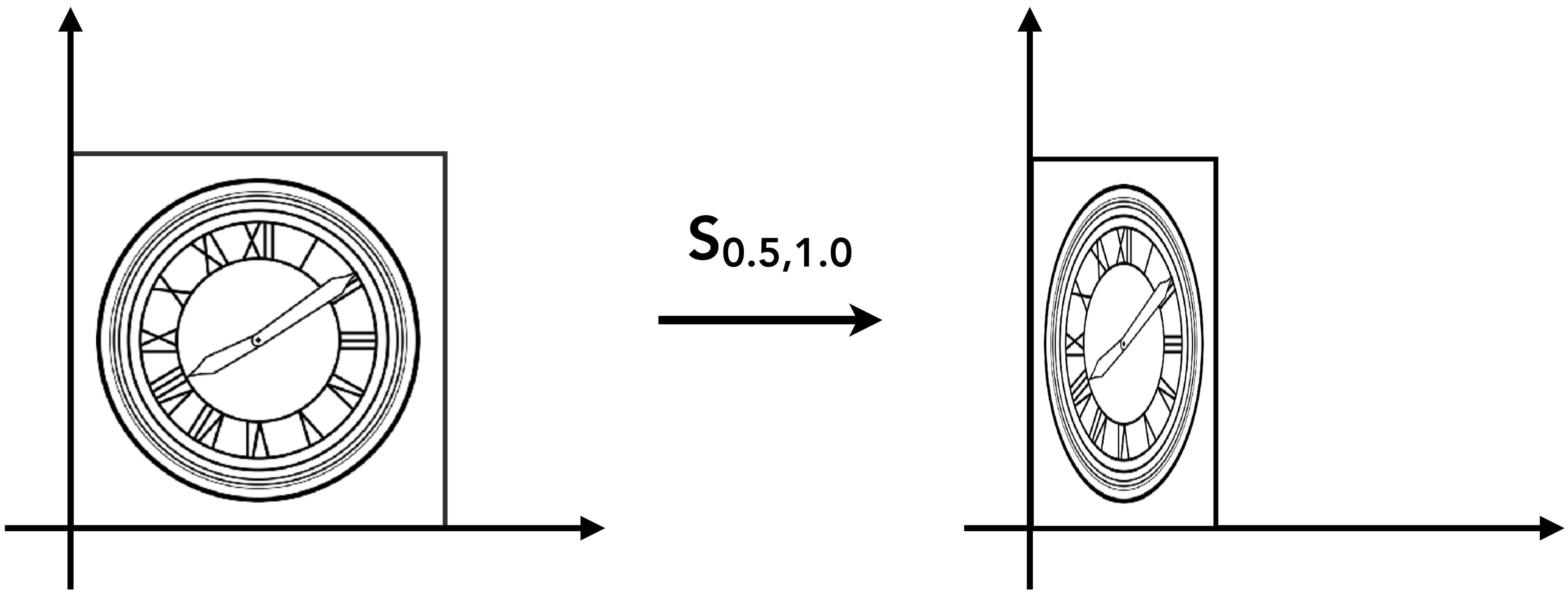
$$y' = sy$$

Scale Matrix



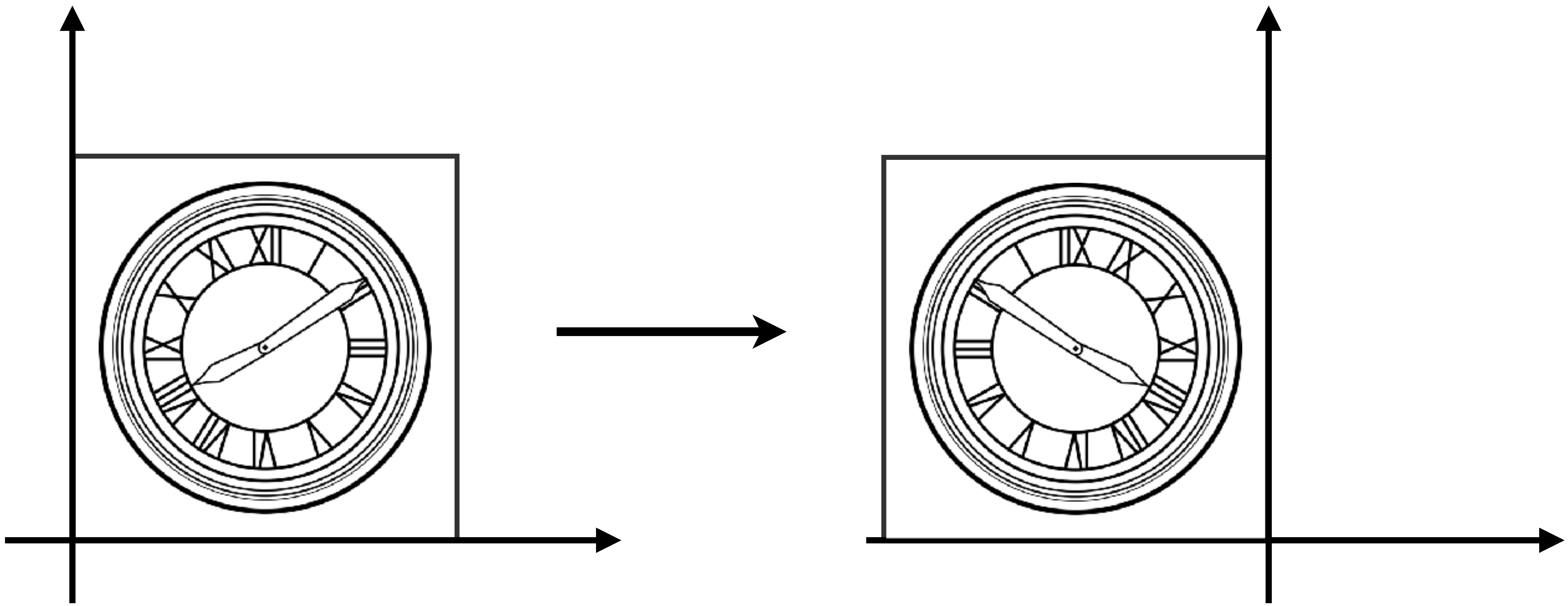
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale (Non-Uniform)



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection Matrix

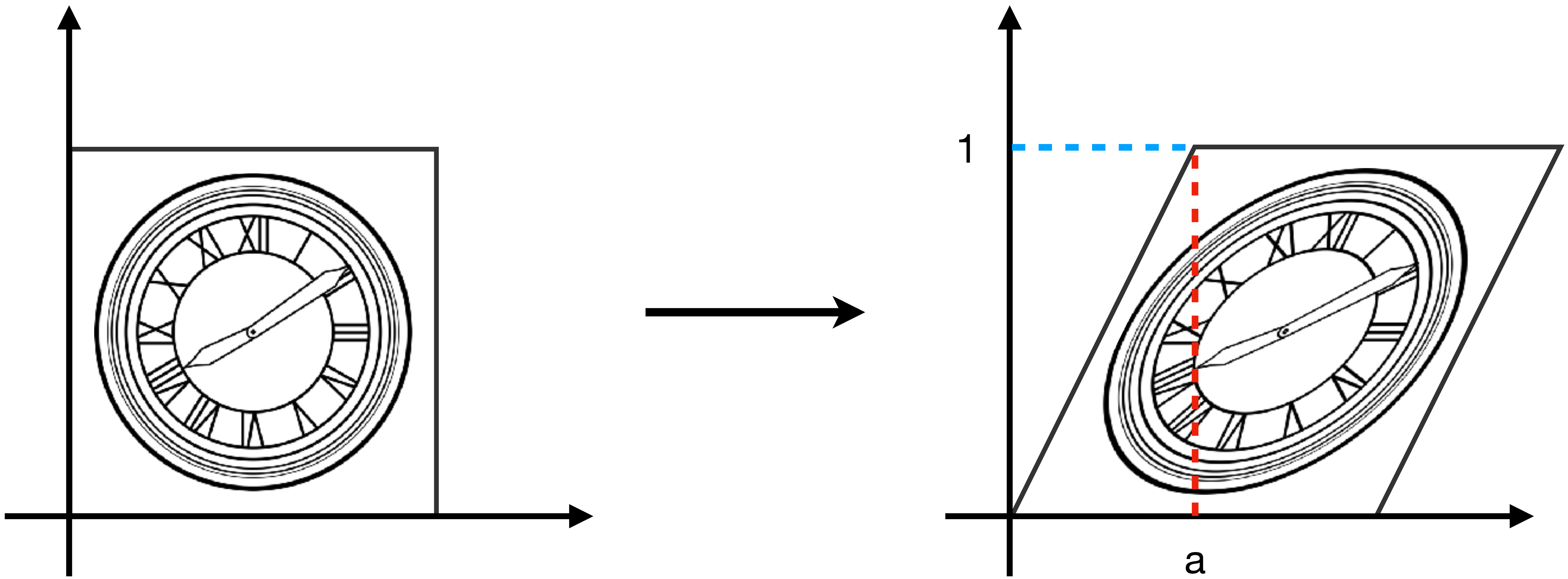


Horizontal reflection:

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear Matrix



Hints:

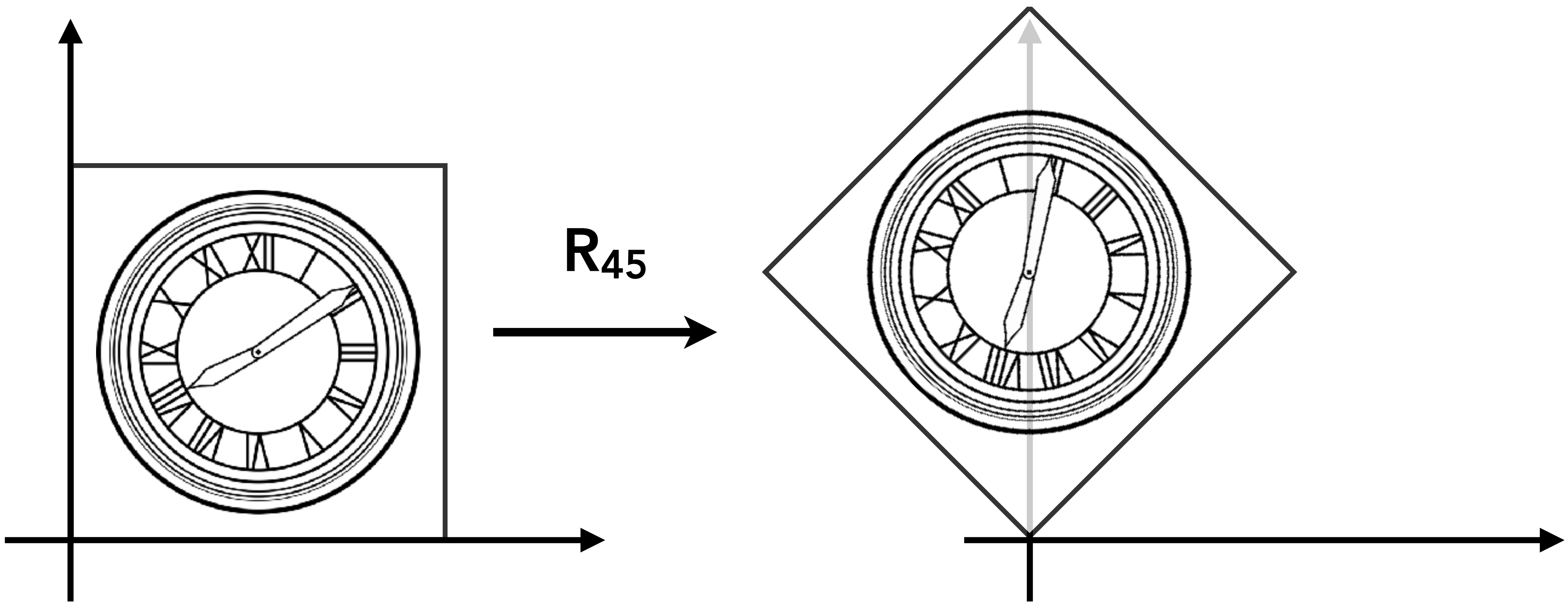
Horizontal shift is 0 at y=0

Horizontal shift is a at y=1

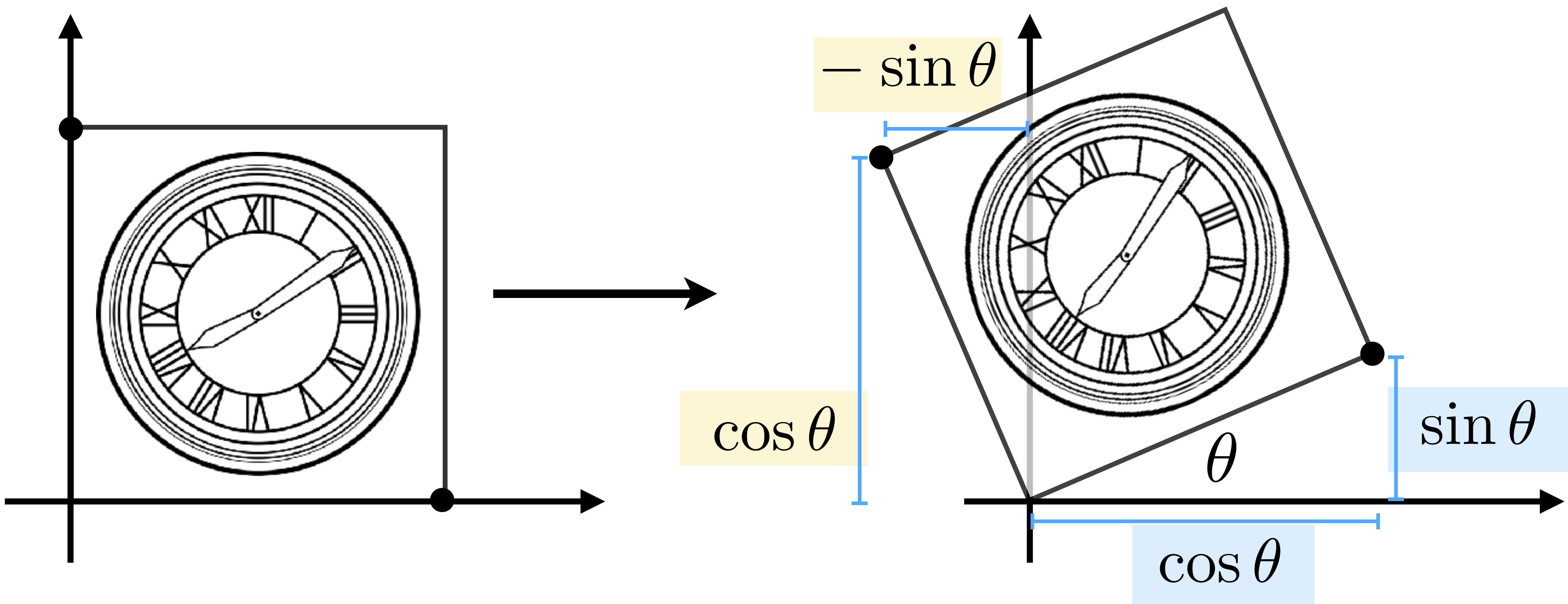
Vertical shift is always 0

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotate (about the origin (0, 0), CCW by default)



Rotation Matrix



$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Linear Transforms = Matrices

(of the same dimension)

$$x' = a\,x + b\,y$$
$$y' = c\,x + d\,y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

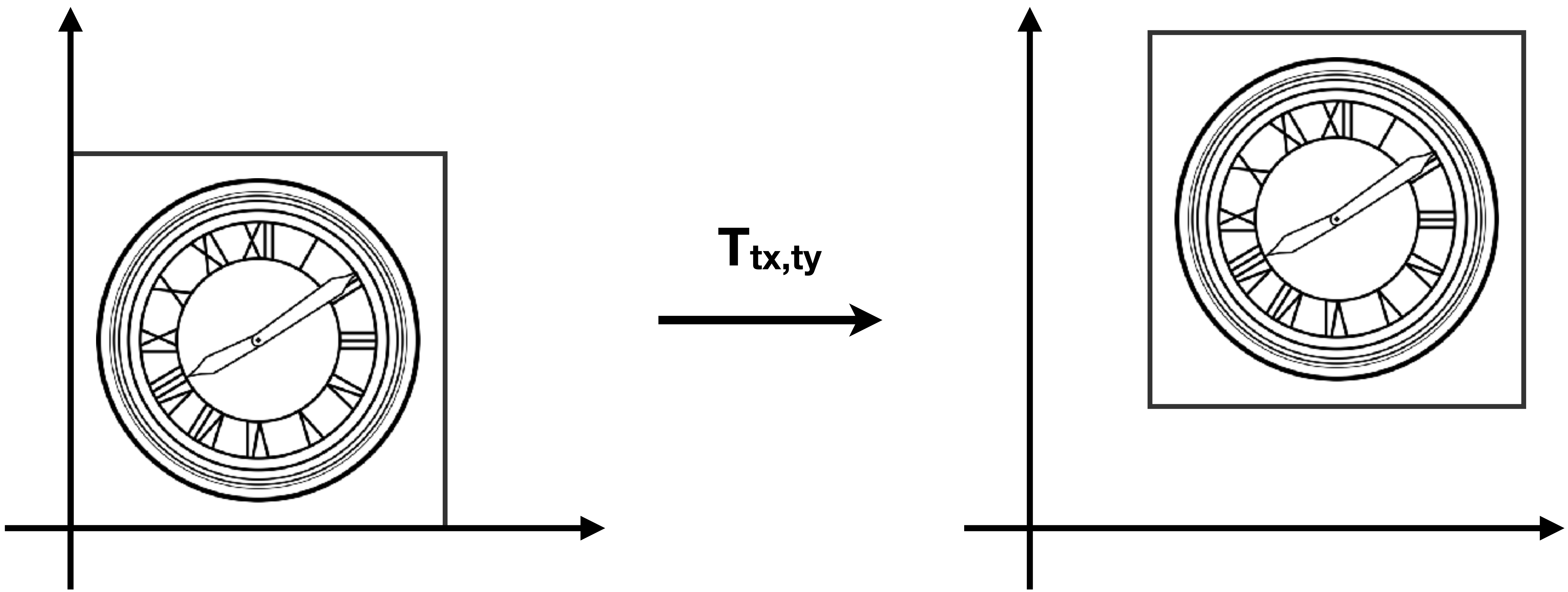
$$\mathbf{x}' = \mathbf{M}\,\mathbf{x}$$

Questions?

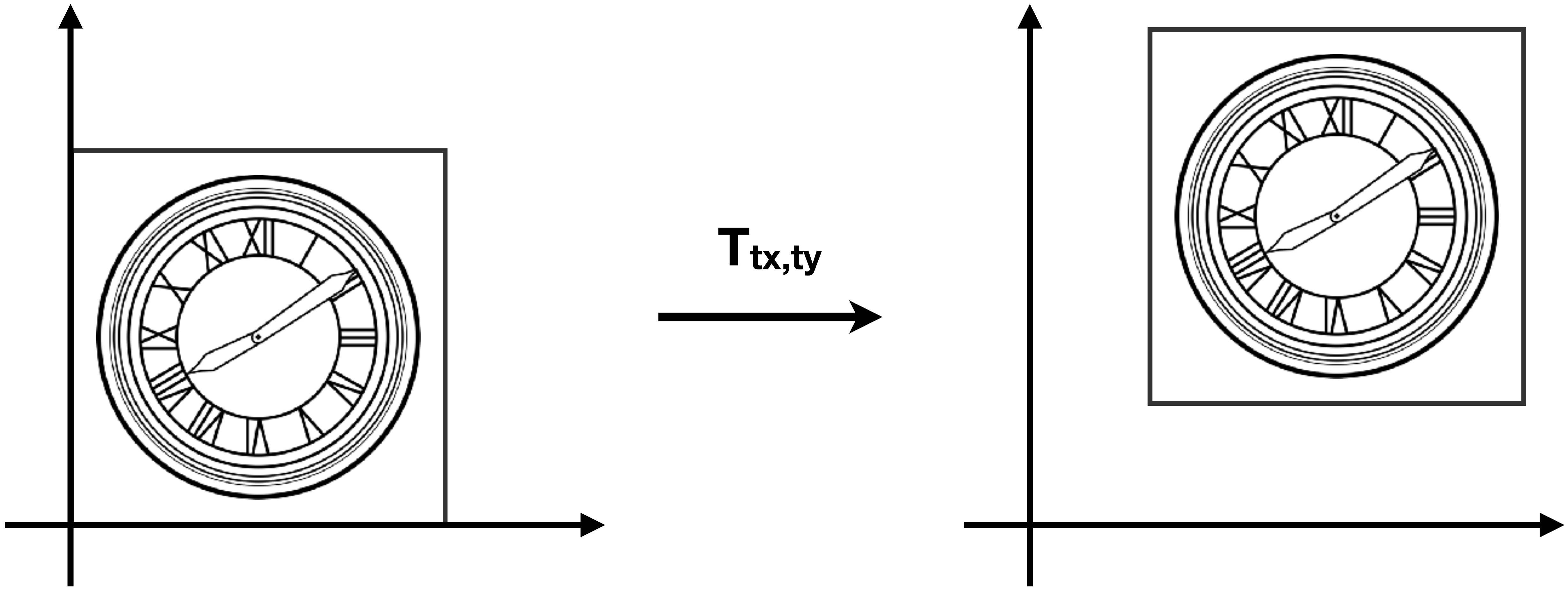
Today

- Why study transformation
- 2D transformations
- Homogeneous coordinates
  - Why homogeneous coordinates
  - Affine transformation

Translation



Translation??



$$x' = x + t_x$$
$$y' = y + t_y$$

Why Homogeneous Coordinates

- Translation cannot be represented in matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

(So, translation is NOT linear transform!)

- But we don't want translation to be a special case

- Is there a unified way to represent all transformations? (and what's the cost?)

Solution: Homogenous Coordinates

Add a third coordinate (*w-coordinate*)

- 2D point = $(x, y, 1)^T$
- 2D vector = $(x, y, 0)^T$

Matrix representation of translations

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

What if you translate a vector?

Homogenous Coordinates

Valid operation if w-coordinate of result is 1 or 0

- vector + vector = vector
- point – point = vector
- point + vector = point
- point + point = ??

In homogeneous coordinates,

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \text{ is the 2D point } \begin{pmatrix} x/w \\ y/w \\ 1 \end{pmatrix}, \ w \neq 0$$

Affine Transformations

Affine map = linear map + translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Using homogenous coordinates:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Transformations

Scale

$$\mathbf{S}(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation

$$\mathbf{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

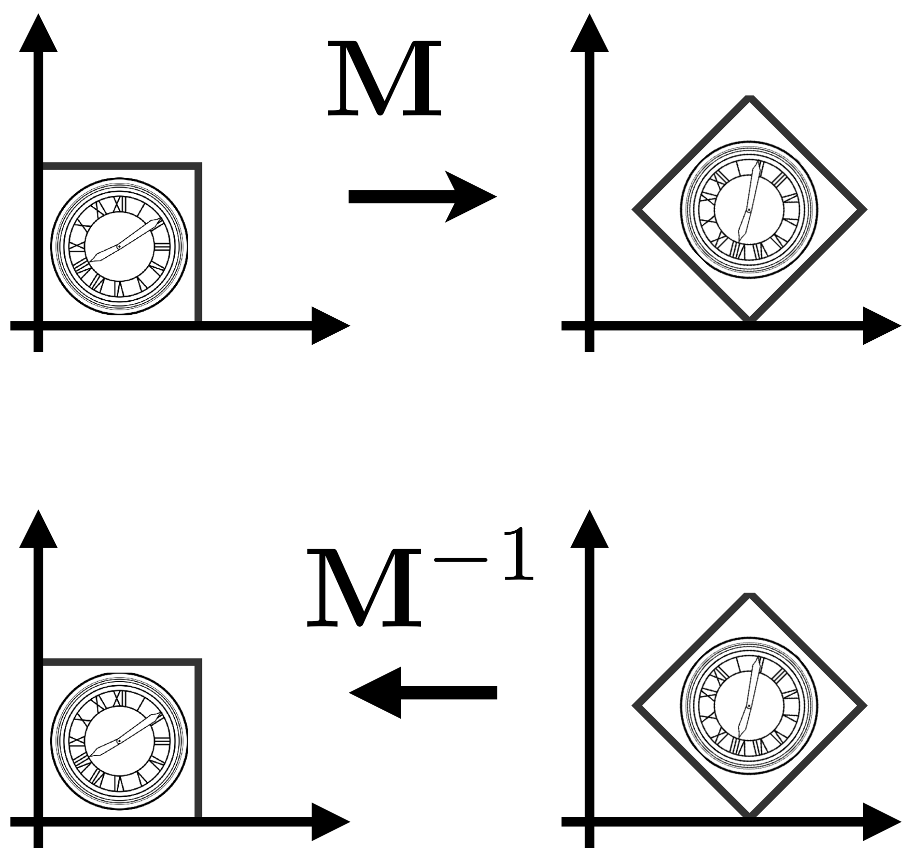
Translation

$$\mathbf{T}(t_x, t_y) = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse Transform

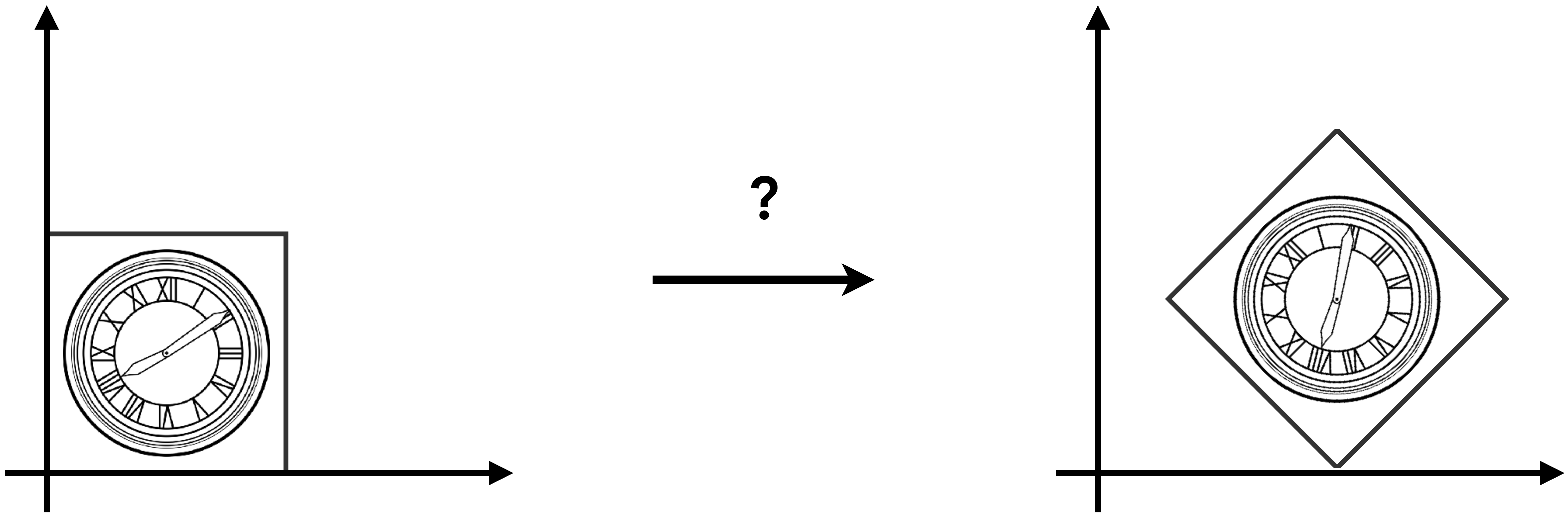
$$\mathbf{M}^{-1}$$

$\mathbf{M}^{-1}$ **is the inverse of transform** $\mathbf{M}$ **in both a matrix and geometric sense**

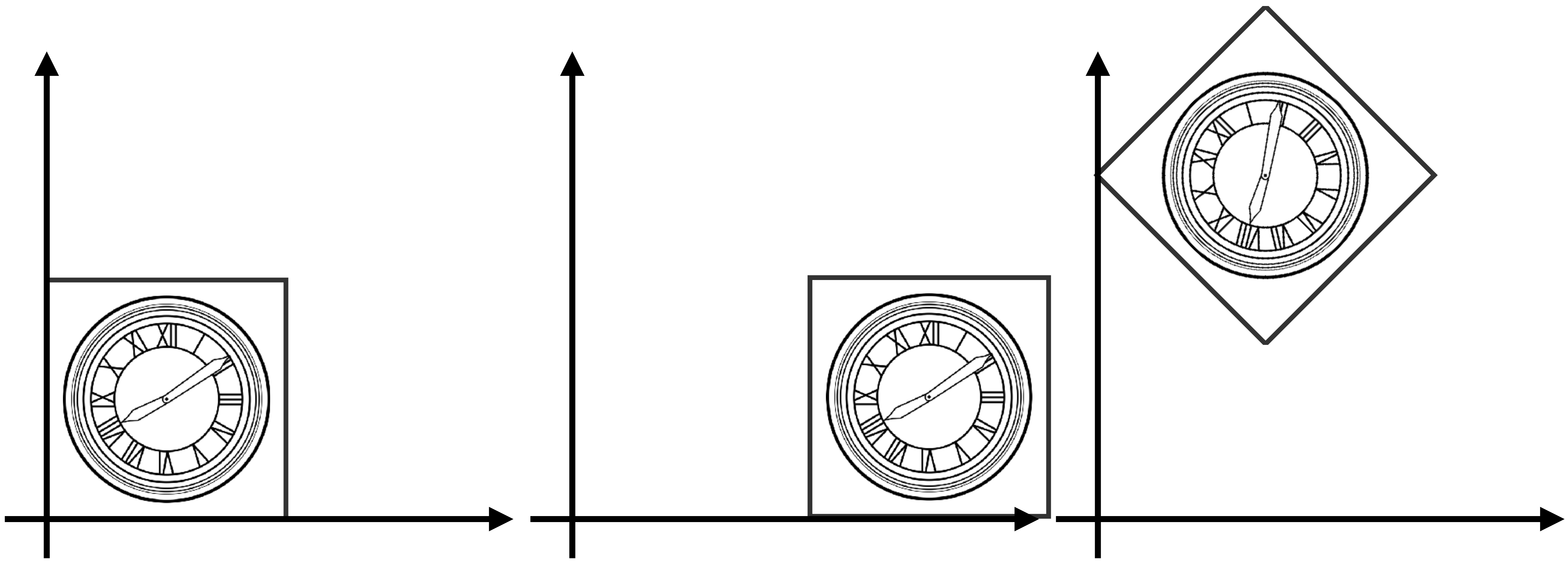


Composing Transforms

Composite Transform

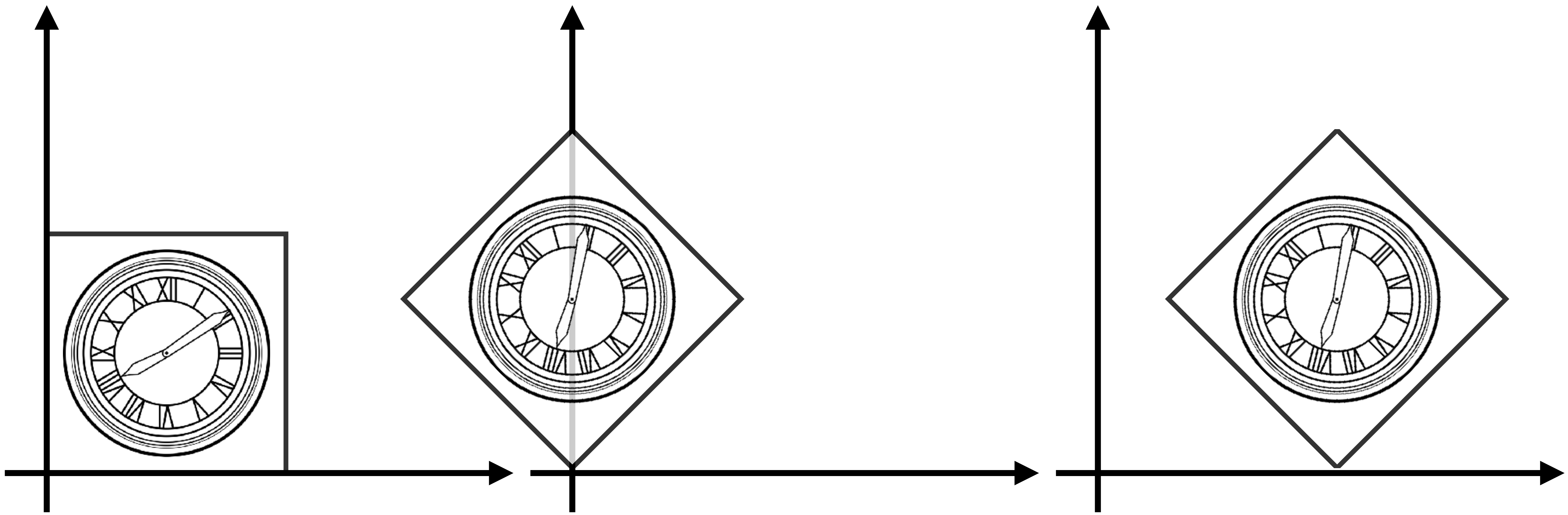


Translate Then Rotate?



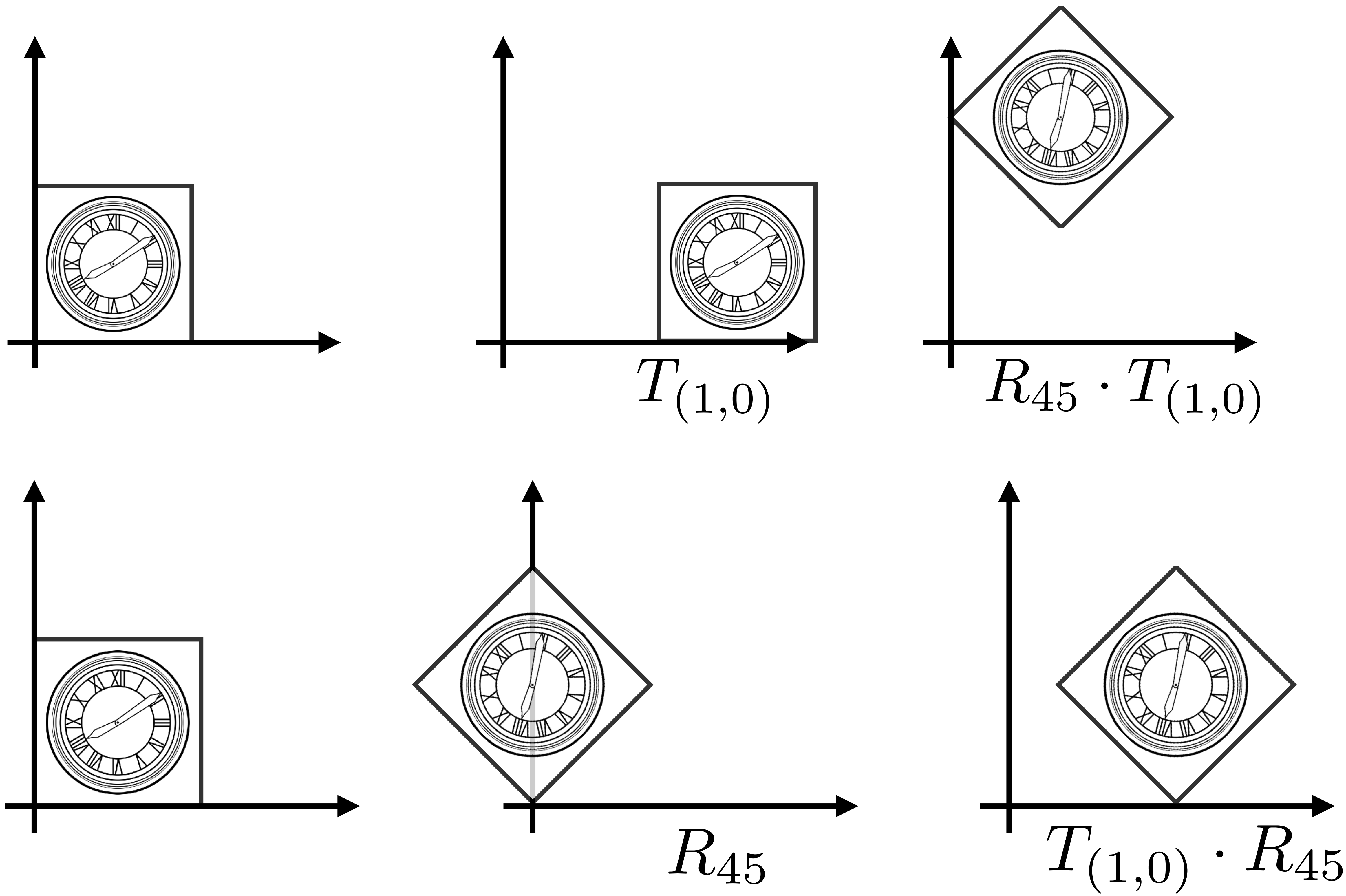
$T_{(1,0)}$ $R_{45} \cdot T_{(1,0)}$

Rotate Then Translate



R_{45} $\qquad$ $T_{(1,0)} \cdot R_{45}$

Transform Ordering Matters!



Transform Ordering Matters!

Matrix multiplication is <u>not</u> commutative

$$R_{45} \cdot T_{(1,0)} \neq T_{(1,0)} \cdot R_{45}$$

Note that matrices are applied right to left:

$$T_{(1,0)} \cdot R_{45} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composing Transforms

Sequence of affine transforms A_1, A_2, A_3, ...

- **Compose by matrix multiplication**
  - **Very important for performance!**

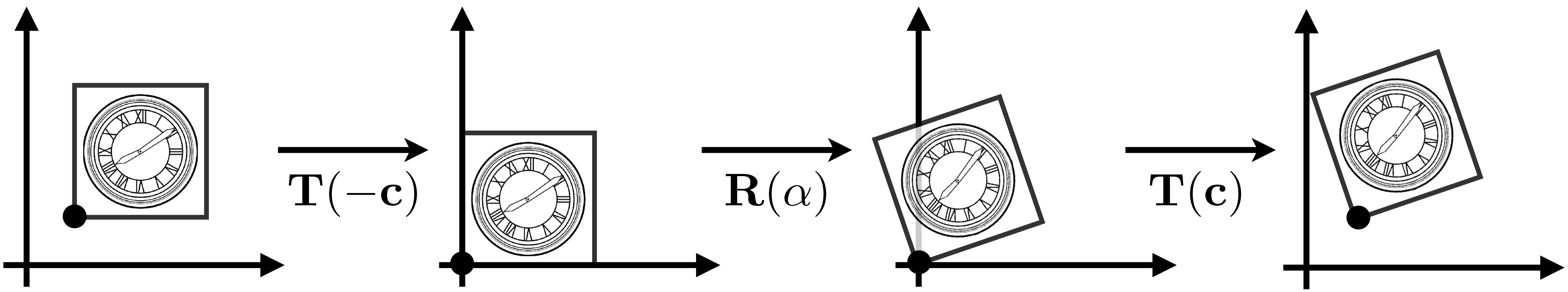
$$A_n(\dots A_2(A_1(\mathbf{x}))) \;=\; \mathbf{A}_n \cdots \mathbf{A}_2 \cdot \mathbf{A}_1 \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Pre-multiply *n* matrices to obtain a single matrix representing combined transform

Decomposing Complex Transforms

How to rotate around a given point c?

1. Translate center to origin
2. Rotate
3. Translate back



Matrix representation?

$$\mathbf{T}(\mathbf{c}) \cdot \mathbf{R}(\alpha) \cdot \mathbf{T}(-\mathbf{c})$$

3D Transforms

3D Transformations

Use homogeneous coordinates again:

- **3D point = $(x, y, z, 1)^T$**
- **3D vector = $(x, y, z, 0)^T$**

In general, (x, y, z, w) ($w \neq 0$) is the 3D point:

$$(x/w, y/w, z/w)$$

3D Transformations

Use 4×4 matrices for affine transformations

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

What's the order?
Linear Transform first or Translation first?

Thank you!

(And thank Prof. Ravi Ramamoorthi and Prof. Ren Ng for many of the slides!)