Skew-symmetric matrices and the Pfaffian

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Abstract. The Pfaffian of the symbols a_{ij} with $i < j$ has a combinatorial interpretation as the signed weight generating function of perfect matchings in the complete graph. By properly specializing the variables, this generating function reduces to the signed weight generating function for the perfect matchings in an arbitrary simple graph.

We construct a weight and sign preserving bijection between two appropriately constructed spaces of permutations: permutations with even cycles and pairs of involutions without fixed points. This bijection gives a purely combinatorial proof that the determinant of a zero axial skew-symmetric matrix is equal to the square of the Pfaffian.

1.0 Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A subset M of edges in G such that no two edges in M is incident upon the same vertex in V is a *matching* in G. If every vertex in V is incident to some edge in M, then the matching is *perfect*.

A perfect matching M in G can be viewed as an involution σ_M without fixed points by interpreting the endpoints of edges in M as forming the 2-cycles of σ_M. We associate a sign to σ_M by taking the parity of the *fundamental transform* of σ_M as defined by Foata [**2**]. To obtain the fundamental transform of σ_M, we write down the cycle decomposition of σ_M so that in each cycle the smallest element comes first, and then the cycles themselves are ordered by increasing smallest element. Erasing the parentheses from this representation of σ_M, one obtains the fundamental transform of σ_M in one line notation.

Next we construct the space PI_{2m} whose elements are ordered pairs (α, β) of involutions without fixed points with weight

$$W(\alpha, \beta) = \prod_{i=1}^{2m} a_{i\alpha_i} \prod_{i=1}^{2m} a_{i\beta_i}$$

The sign of the pair (α, β) is the product of the parities of the fundamental transforms of α and β.

Let E_{2m} denote the collection of permutations on $2m$ symbols whose cycle decomposition contains only cycles of even lengths. E_{2m} is turned into a weighted, signed space by setting for each $\sigma \in E_{2m}$

$$w(\sigma) = \prod_{i=1}^{2m} a_{i\sigma_i}, \quad sign(\sigma) = (-1)^{i(\sigma)+f(\sigma)}$$

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where $i(\sigma)$ and $f(\sigma)$ are the number of inversions and the number of falls of σ, respectively.

In this paper we construct a weight and sign preserving bijection between PI_{2m} and E_{2m}. This bijection provides a purely combinatorial proof that the square of the Pfaffian of the symbols a_{ij} is the determinant of the corresponding zero axial skew-symmetric matrix in the a_{ij}'s.

Algebraic proofs of the above result about the Pfaffian were given by Pfaff [7], Jacobi [4], Cayley [1] and Horner[3]. A detailed account of their individual contributions can be found in the comprehensive treatise by Muir [5].

Tutte [8] gave a graph theoretic interpretation of zero axial skew-symmetric matrices by relating the determinant to perfect matchings. The skew-symmetric matrix constructed from the incidence matrix of a simple graph G is thus sometimes referred to as the *Tutte matrix* of G.

Recently, there has been renewed interest in Tutte's interpretation both for theoretical and computational reasons. This is a consequence of the fact that the determinant can be evaluated fast in parallel time. For this and related current results on matchings, see [6].

1.1 Preliminaries

Let D_n denote the determinant of the $n \times n$ zero axial skew-symmetric matrix in the indeterminates a_{ij}, $i < j$. Thus we have for instance,

$$D_4 = \det \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}$$

S_n denotes the set of permutations in n symbols.

Expanding the determinant from the first principles and noting that any subdiagonal term picked by a permutation carries a negative sign, we find that

$$D_n = \sum_{\sigma \in S_n} (-1)^{i(\sigma)} (-1)^{f(\sigma)} \prod_{i=1}^{n} a_{i\sigma_i} \qquad (1.1)$$

where $f(\sigma)$ denotes the number of *falls* of σ: i.e. the number of indices i for which $i > \sigma_i$. In (1.1) we adopt the convention that $a_{i\sigma_i} = a_{\sigma_i i}$ and $a_{ii} = 0$.

Note that any permutation with a fixed point contributes zero to the sum in (1.1). Furthermore, if $\sigma \in S_n$ has an odd cycle of length > 1, then we can invert the unique odd cycle of σ which contains the smallest index in $\{1, 2, \cdots, n\}$ while keeping the other cycles unaltered. The resulting permutation has the same weight as σ but opposite sign. Since this correspondence is bijective, the terms arising from such permutations cancel out in pairs in (1.1). In particular D_n vanishes for odd n.

We summarize this observation as a lemma:

Lemma 1.1.

$$D_{2m} = \sum (-1)^{i(\sigma)+f(\sigma)} \prod_{i=1}^{n} a_{i\sigma_i} = \sum sign(\sigma) w(\sigma) \qquad (1.2)$$

where the summation is over all permutations in S_{2m} with cycles of even lengths.

Our point of departure for the combinatorial arguments that follow will be the expression (1.2) for D_{2m}.

1.2 Perfect matchings and the Pfaffian

Cayley [1] proved that $D_{2m} = P_{2m}^2$ is a perfect square. P_{2m} is the *Pfaffian* of the indeterminates a_{ij}. For example

$$D_4 = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 = P_4^2.$$

There is a simple combinatorial rule due to Pfaff [7] to produce the monomials together with their signs that appear in P_{2m}. This can be stated as follows:

Let σ be an involution in S_{2m} without fixed points. We write σ as a disjoint product of its cycles so that in each 2-cycle the smaller element comes first, and the cycles themselves are ordered by increasing smallest element. This is called the *canonical form* of σ. For example

$$\sigma = (17)(28)(34)(56) \qquad (1.3)$$

is in canonical form.

Dropping the parentheses from the canonical form of σ gives a permutation in S_8 whose structure in one line notation is 17283456. This is the fundamental transform of σ. Since sorting of the cycles of σ is involved in the construction of the fundamental transform, we denote the resulting permutation by $sort(\sigma)$.

Note that σ can be interpreted as a coding for the monomial

$$m(\sigma) = a_{17}a_{28}a_{34}a_{56}.$$

The contribution of σ to the Pfaffian P_8 is the signed monomial

$$(-1)^{i(sort(\sigma))} m(\sigma) = -a_{17}a_{28}a_{34}a_{56}.$$

Pfaff argued that all the monomials in P_{2m} in general are obtained in this manner.

Thus the number of different monomials in P_{2m} is the number of involutions in S_{2m} without fixed points: i.e. $1 \cdot 3 \cdot 5 \cdots (2m-1)$. Furthermore, every such σ can be interpreted as a coding for a perfect matching in the complete graph K_{2m}. For instance the permutation σ of (1.3) corresponds to the perfect matching in K_8 as shown in Figure 1.

Note that each perfect matching actually carries a sign in the monomial interpretation.

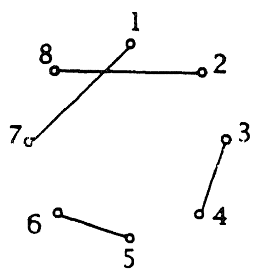

Figure 1

1.3 The bijection

Let E_{2m} denote the collection of permutations in S_{2m} with even cycles. We turn E_{2m} into a signed, weighted space by setting

$$sign(\sigma) = (-1)^{i(\sigma)+f(\sigma)}, \quad w(\sigma) = \prod_{i=1}^{2m} a_{i\sigma_i}$$

for every $\sigma \in E_{2m}$. By Lemma 1.1, the weight generating function of E_{2m} is the determinant D_{2m}.

Now let I_{2m} denote the set of involutions without fixed points in S_{2m}, and put $PI_{2m} = I_{2m} \times I_{2m}$. We also turn PI_{2m} into a signed, weighted space by setting for each pair $(\alpha, \beta) \in PI_{2m}$

$$sign(\alpha,\beta) = (-1)^{i(sort(\alpha))} \cdot (-1)^{i(sort(\beta))}, \quad W(\alpha,\beta) = m(\alpha) \cdot m(\beta).$$

Clearly,

$$\sum_{(\alpha,\beta)\in PI_{2m}} sign(\alpha,\beta) W(\alpha,\beta)$$

$$= \Big[\sum_{\alpha\in I_{2m}} (-1)^{i(sort(\alpha))} m(\alpha)\Big]\Big[\sum_{\beta\in I_{2m}} (-1)^{i(sort(\beta))} m(\beta)\Big] = P_{2m}^2.$$

Thus to give a combinatorial proof of the identity $D_{2m} = P_{2m}^2$, it suffices to construct a bijection Φ between the two spaces E_{2m} and PI_{2m} which is both weight and sign preserving. This bijection Φ is constructed as follows:

Starting with an element $\sigma \in E_{2m}$, we first draw the cycle diagram of σ. For instance the cycle diagram of the permutation $\sigma = (1\ 12\ 9\ 7\ 2\ 4)(3\ 5)(6\ 10\ 11\ 8)$ appears in Figure 2.

Now in each cycle of σ we locate the smallest element. Note that precisely one of the edges incident to the smallest element is a fall of σ. We label this edge by α in each cycle of σ and then assign the labels β and α alternately to all the other edges. This is always possible since the cycles of σ are all of even lengths. For the

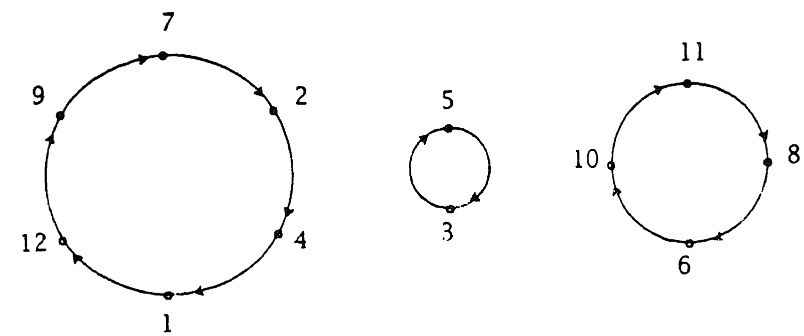

Figure 2

permutation in Figure 2 this results in the assignments in Figure 3. and defines an element of PI_{2m} by interpreting the labels of the edges marked α as the 2-cycles of an involution without fixed points α, and the labels of the edges marked β as the 2-cycles of an involution without fixed points β. For the example σ; this yields the pair (α, β) where

$$\alpha = (1\ 4)(2\ 7)(3\ 5)(6\ 8)(9\ 12)(10\ 11),$$
$$\beta = (1\ 12)(2\ 4)(3\ 5)(6\ 10)(7\ 9)(8\ 11).$$

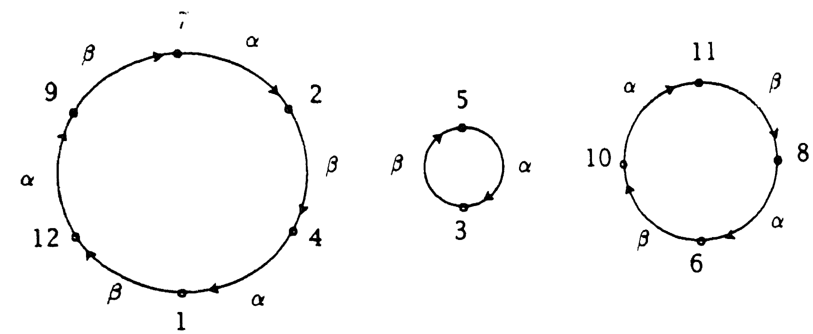

Figure 3

Conversely, given a pair $(\alpha, \beta) \in PI_{2m}$, we draw an edge labeled α between the vertices i and j of the null graph for every 2-cycle (ij) of α, and draw an edge labeled β connecting the vertices r and s for every 2-cycle (rs) of β. It is not difficult to see that the resulting graph decomposes into a union of (undirected) cycles of even lengths with alternating labels. Since exactly one of the edges incident to the smallest element in each such cycle is labeled α, we can direct this edge toward the smallest element. This gives a direction to each cycle and defines a permutation σ in E_{2m} such that $\Phi(\sigma) = (\alpha, \beta)$. For instance if

$$\alpha = (1\ 2)(3\ 6)(4\ 11)(5\ 10)(7\ 9)(8\ 12),$$
$$\beta = (1\ 10)(2\ 5)(3\ 12)(4\ 7)(6\ 8)(9\ 11),$$

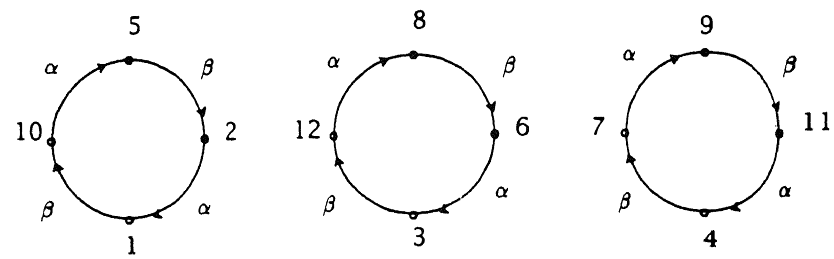

Figure 4

then $\Phi^{-1}(\alpha, \beta)$ has the cycle diagram in Figure 4.

Theorem. *The mapping Φ is a weight and sign preserving bijection between the spaces E_{2m} and PI_{2m}.*

Proof: It is trivial to see that Φ is a weight preserving bijection.

Next we show that Φ also preseves sign. This turns out to be slightly more difficult. We proceed in two steps and prove two auxilary lemmas.

Lemma 1.2. *Suppose $\sigma \in E_{2m}$ and that σ' is obtained from σ by interchanging two adjacent symbols in a cycle of σ. Then $sign(\sigma) = sign(\sigma')$ if and only if $sign(\Phi(\sigma)) = sign(\Phi(\sigma'))$.*

Proof: Suppose that the symbols interchanged in σ are x and y. Clearly if x and y form a 2-cycle of σ then $\sigma' = \sigma$ and we are done.

Hence we may assume that x and y are contained in a larger cycle of σ. Then without loss of generality, we have the following situation:

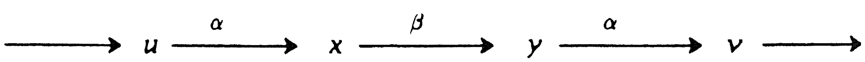

Figure 5

We will denote this portion of σ without writing the arrows in Figure 5 by $uxyv$. Note that the exchange has no effect on the involution β so that if $\Phi(\sigma) = (\alpha, \beta)$, then $\Phi(\sigma') = (\alpha', \beta)$.

Furthermore, the cycle structure of σ' is identical to that of σ. Thus we are only concerned here with the relation between the number of falls of σ and the number of falls of σ', and the corresponding effect this interchange produces in the parities of $sort(\alpha)$ and $sort(\alpha')$. It is clear that the change in the number of falls of σ after the interchange depends locally on the relative magnitudes of the four numbers u, x, y, v.

Let $\{u, x, y, v\} = \{a, b, c, d\}$ where $a < b < c < d$. Note that $uxyv$ can be any one of the 24 permutations of $\{a, b, c, d\}$.

If $uxyv$ happens to be $abcd$ for instance, this will be paired with $acbd$ after the interchange. The difference between the number of falls of $abcd$ and those of

$acbd$ is an odd number, which means that $sign(\sigma') = -sign(\sigma)$. We will write in short

$$abcd \longleftrightarrow acbd(-).$$

to summarize this fact. On the other hand, the above interchange in σ corresponds to removing the transpositions $(ab), (cd)$ from α and replacing them by the transpositions $(ac), (bd)$.

With the above notation, the twelve possible pairings resulting from exchanging x and y together with the corresponding changes in the 2-cycles of α are given below:

1)	$acbd \longleftrightarrow adcb\ (-)$	$(ac),(bd) \longleftrightarrow (ad),(bc)$
2)	$bcda \longleftrightarrow bdca\ (-)$	$(ad),(bc) \longleftrightarrow (ac),(bd)$
3)	$cabd \longleftrightarrow cbad\ (-)$	$(ac),(bd) \longleftrightarrow (ad),(bc)$
4)	$dabc \longleftrightarrow dbac\ (-)$	$(ad),(bc) \longleftrightarrow (ac),(bd)$
5)	$abdc \longleftrightarrow adbc\ (+)$	$(ab),(cd) \longleftrightarrow (ad),(bc)$
6)	$bacd \longleftrightarrow bcad\ (+)$	$(ab),(cd) \longleftrightarrow (ad),(bc)$
7)	$cbda \longleftrightarrow cdba\ (+)$	$(ad),(bc) \longleftrightarrow (ab),(cd)$
8)	$dacb \longleftrightarrow dcab\ (+)$	$(ad),(bd) \longleftrightarrow (ab),(cd)$
9)	$abcd \longleftrightarrow acbd\ (-)$	$(ab),(cd) \longleftrightarrow (ac),(bd)$
10)	$badc \longleftrightarrow bdac\ (-)$	$(ab),(cd) \longleftrightarrow (ac),(bd)$
11)	$cadb \longleftrightarrow cdab\ (-)$	$(ac),(bd) \longleftrightarrow (ab),(cd)$
12)	$dbca \longleftrightarrow dcba\ (-)$	$(ac),(bd) \longleftrightarrow (ab),(cd)$

All of these twelve cases can be summarized by the following diagram:

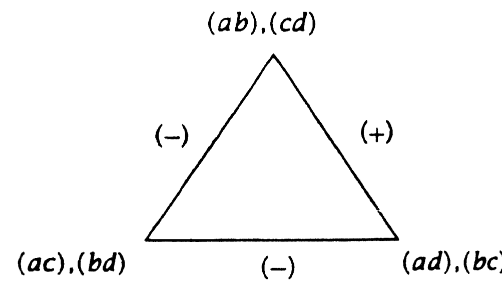

Figure 6

We need to show that if the pairs of 2-cycles are exchanged as indicated in Figure 6, then the signs of the fundamental transforms of the resulting involutions are related by the label on the corresponding edge. In the exchange $(ac), (bd) \longleftrightarrow (ad), (bc)$ for example, the smallest element in each 2-cycle is preserved. Thus when the sorting of the transpositions of α and α' are carried out, these pairs of transpositions will be placed in exactly the same relative locations, since the sorting is done according to the smallest elements in each cycle. Hence in these cases

$sort(\alpha')$ is obtained from $sort(\alpha)$ by interchanging only the elements c and d. But this means $sort(\alpha')$ and $sort(\alpha)$ have opposite parity as required.

If the exchanged pairs are $(ab),(cd) \longleftrightarrow (ac),(bd)$, we consider the canonical forms of α and α'. Since $a < b < c < d$, these are of the form

$$\alpha = \cdots (ab)(i_1 i_2) \cdots (i_{r-1} i_r)(r_{r+1} i_{r+2}) \cdots (i_{k-1} i_k)(cd) \cdots$$
$$\alpha' = \cdots (ac)(i_1 i_2) \cdots (i_{r-1} i_r)(bd)(r_{r+1} i_{r+2}) \cdots (i_{k-1} i_k) \cdots$$

with $a < i_1 < \cdots < i_{r-1} < b < i_{r+1} < \cdots < i_{k-1} < c$, and where possibly $r = k$. But now we see that in $sort(\alpha)$, the symbols c and b can be interchanged with an odd number of adjacent transposition of symbols, and then after that the pairs $i_{k-1} i_k$ through $i_{r+1} i_{r+2}$ can be moved to the right of (bd) with an even number of adjacent interchanges. Thus $sort(\alpha')$ is obtained from $sort(\alpha)$ by an odd number of adjacent interchanges. Since each such interchange changes the parity of the underlying permutation, $sort(\alpha')$ and $sort(\alpha)$ have opposite parity. The proof of the remaining case is similar and will be omitted. ▮

Note that Lemma 1.2 has the following consequence. If $\sigma \in E_{2m}$ and σ' is obtained from σ by permuting the symbols in each cycle of σ arbitrarily, then

$$sign(\sigma) = sign(\sigma') \quad \text{if and only if } sign(\Phi(\sigma)) = sign(\Phi(\sigma')). \qquad (1.4)$$

This is because each such permutation can be realized as a sequence of interchanges of adjacent symbols in the cycles of σ and Lemma 1.2 is applicable at each step.

Let us put $\sigma \sim \sigma'$ if one is obtained from the other by a permutation that may move only the symbols within individual cycles. Then $\sim$ is an equivalence relation on E_{2m} and as such, breaks E_{2m} up into equivalence classes $\theta_1, \theta_2, \cdots, \theta_d$. By 1.4, the bijection Φ restricted to each equivalence class θ_i is either sign preserving on all of θ_i or sign reversing on all of θ_i, $i = 1, 2, \cdots, d$.

To show that Φ is sign preserving, it suffices then to show that $sign(\pi) = sign(\Phi(\pi))$ for a selected permutation $\pi = \pi(\theta_i)$ in each class. We select this representative π to be the unique permutation in the given class with precisely one fall in each cycle. This amounts to rearranging the symbols in each cycle in increasing order, starting from the smallest element. The unique fall in each cycle of π is caused by the edge connecting the largest element in the cycle to the smallest.

For example, the representative π of the equivalence class containing the permutation σ depicted in Figure 2 has the following cycle diagram given in Figure 7.

Lemma 1.3. *For each representative permutation* $\pi = \pi(\theta_i)$ *of an equivalence class* θ_i *of* E_{2m} *we have*

$$sign(\pi) = sign(\Phi(\pi)).$$

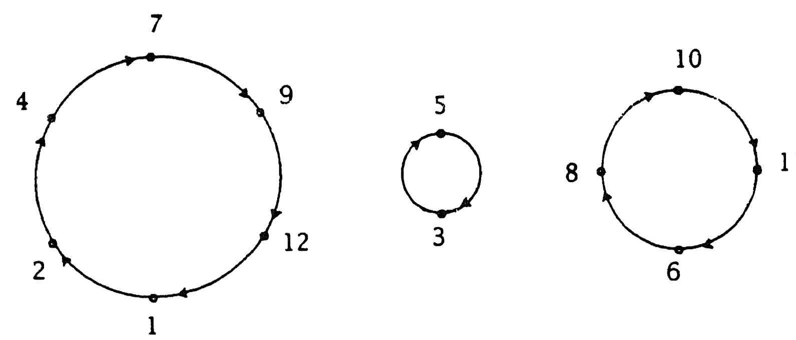

Figure 7

Proof: Note that any $\sigma \in E_{2m}$ with a total number of k cycles has parity $(-1)^k$ since each cycle of σ is an odd permutation. By construction, π has exactly one fall in each cycle, and therefore we always have

$$sign(\pi) = (-1)^{i(\pi)+f(\pi)} = (-1)^{2f(\pi)} = +1.$$

Thus we need to show that $sign(\Phi(\pi)) = +1$.

Suppose $\Phi(\pi) = (\alpha, \beta)$. From a cycle of π that contains the symbols $a_1 < a_2 < \cdots < a_{2r}$, the bijection Φ contributes the 2-cycles

$$(a_1 a_{2r})(a_2 a_3) \cdots (a_{2r-2} a_{2r-1})$$

to α and the 2-cycles

$$(a_1 a_2)(a_3 a_4) \cdots (a_{2r-1} a_{2r})$$

to β.

Suppose the cycles of π contain the symbols

$$a_1 < a_2 < \cdots < a_{2r}, \quad b_1 < b_2 < \cdots < b_{2s}, \quad c_1 < c_2 < \cdots < c_{2t}, \cdots$$

in some ordering of the cycles, say by increasing smallest element. Then

$$\alpha = (a_1 a_{2r})(a_2 a_3) \cdots (a_{2r-2} a_{2r-1})(b_1 b_{2s})(b_2 b_3) \cdots (b_{2s-2} b_{2s-1})(c_1 c_{2t})(c_2 c_3) \cdots (c_{2t-2} c_{2t-1}) \cdots \qquad (1.5)$$

and

$$\beta = (a_1 a_2)(a_3 a_4) \cdots (a_{2r-1} a_{2r})(b_1 b_2)(b_3 b_4) \cdots (b_{2s-1} b_{2s})(c_1 c_2)(c_3 c_4) \cdots (c_{2t-1} c_{2t}) \cdots \qquad (1.6)$$

where α and β are not necessarily in canonical form.

Let $drop(\alpha)$ and $drop(\beta)$ denote the two permutations in one line notation obtained by dropping the parentheses from the expressions for α and β given in (1.5) and (1.6) , respectively.

Note that the number of inversions in $drop(\alpha)$ among the symbols a_i themselves is even, and the number of inversions in $drop(\beta)$ among the symbols a_i is also even (it is zero for $drop(\beta)$). Similarily, number of inversions among the symbols b_i is even in both $drop(\alpha)$ and $drop(\beta)$; the number of inversions among the symbols c_i is even in both $drop(\alpha)$ and $drop(\beta)$ etc.

Let now $i_{a,b}(drop(\alpha))$ denote the number of inversions in $drop(\alpha)$ among the various symbols a_i and b_j, and let $i_{a,b}(drop(\beta))$ denote the number of inversions in $drop(\beta)$ among the various symbols a_i and b_j. We have

$$i_{a,b}(drop(\alpha)) = i_{a,b}(drop(\beta)) = \#\{(a_i, b_j) | a_i > b_j\}.$$

In particular $i_{a,b}(drop(\alpha)) + i_{a,b}(drop(\beta))$ is an even number. Similarily, we have that $i_{a,b}(drop(\alpha)) + i_{a,b}(drop(\beta))$, $i_{b,c}(drop(\alpha)) + i_{b,c}(drop(\beta))$, etc. are all even numbers. This immediately implies that

$$(-1)^{i(drop(\alpha))} \cdot (-1)^{i(drop(\beta))} = +1.$$

To prove the lemma we observe that $sort(\alpha)$ is obtained from $drop(\alpha)$ by sliding various *pairs* of symbols in $drop(\alpha)$ past one another during the process of sorting (i. e. an even number of transpositions is required) so that $sort(\alpha)$ and $drop(\alpha)$ have the same parity. ▮

Combining lemmas 1.2 and 1.3, we have that the bijection Φ we have constructed is also sign preserving, and this completes the proof of the theorem. ▮

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Construction methods for adjusted orthogonal row-column designs

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Abstract. Adjusted orthogonal row-column designs have certain desirable properties. In this paper we give a definition of adjusted orthogonal row-column designs, summarise the known designs, give some construction methods and indicate some open problems. We briefly consider the relationship between adjusted orthogonal row-column designs and orthogonal main effects block designs.

1. Introduction

We begin with some definitions, based on notation given in Preece (1976). Consider a (t, b, r, k, λ) BIBD. It has two *constraints*, namely the blocks and the treatments, which occur at b and t levels, respectively. A Latin square design has three constraints — the rows, the columns and the treatments — and a pair of mutually orthogonal Latin squares have 4 constraints — the rows, the columns, and the two sets of treatments. After ordering a design's constraints, we can define the *incidence matrix* of the x^{th} constraint with respect to the y^{th} by

$$N_{xy} = (n_{ij}),$$

where n_{ij} is the number of times that the i^{th} level of the x^{th} constraint occurs with the j^{th} level of the y^{th} constraint. N_{xy} is a $k_x \times k_y$ matrix, where there are k_i levels of the i^{th} constraint, and $N_{yx} = N_{xy}^T$.

For example, if we regard the blocks, of a BIBD, as the first constraint and the treatments as the second constraint, then N_{21} is the usual incidence matrix and

$$N_{21} N_{21}^T = (r - \lambda) I + \lambda J.$$

A row-column design is a design with 3 constraints, namely rows, with k levels, columns, with b levels, and treatments, with t levels. Each cell in the $k \times b$ array contains exactly one treatment. The matrices N_{13} and N_{23} are the incidence

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