A Combinatorial Generalization of a Putnam Problem

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As a part of the thirty-fourth William Lowell Putnam Mathematical Competition, the following problem appeared in the Monthly [2]:

Let \( a_1, a_2, \ldots, a_{2n+1} \) be a sequence of integers such that, if any of them is removed, the remaining ones can be divided into two sets of \( n \) integers with equal sums. Prove \( a_1 = a_2 = \cdots = a_{2n+1} \).

Here we give a combinatorial proof of a generalization of this problem. The arguments rely on a matrix theoretic formulation of the original problem and elementary properties of cyclotomic polynomials.

**Theorem 1** Let \( \xi \) be a primitive \( q \)-th root of unity where \( q = p^r \), \( p \) prime. Suppose we are given a sequence \( S \) of \( qn + 1 \) complex numbers \( z_1, z_2, \ldots, z_{qn+1} \) with the property that for every \( i \), \( 1 \leq i \leq qn + 1 \), \( S \setminus \{z_i\} \) can be partitioned into \( q \) equal size subsets \( S_{i,0}, S_{i,1}, \ldots, S_{i,q-1} \) with

\[
\sum_{k=0}^{q-1} \xi^k \sum_{z_j \in S_{i,k}} z_j = 0. \tag{1}
\]

Then \( z_1 = z_2 = \cdots = z_{qn+1} \).

Note that the original problem is a special case of Theorem 1 in which \( p = 2 \), \( r = 1 \) and each \( z_i \) is an integer.

**Proof** For each \( i \) fix a partition \( S_{i,0}, S_{i,1}, \ldots, S_{i,q-1} \) of \( S \setminus \{z_i\} \) satisfying (1). Let \( N = qn \) and consider the \( (N + 1) \times (N + 1) \) zero diagonal matrix \( A = \|a_{ij}\| \) where for \( i \neq j \), \( a_{ij} = \xi^k \) if and only if \( z_j \in S_{i,k} \). If we put \( \mathbf{z} = [z_1, z_2, \ldots, z_{N+1}]^T \), then \( \mathbf{z} \) is a solution of the linear system \( A \mathbf{z} = \mathbf{0} \).
Since \(\sum_{k=0}^{q-1} \xi^k = 0\), \(A\) is singular with zero row sums and \([1,1,\ldots,1]^T\) is in the kernel of \(A\). Thus to prove the theorem, it suffices to show that \(\text{rank}(A) = N\).

Let \(f(x) \mid_{x^k}\) denote the coefficient of the term \(x^k\) in a polynomial \(f(x)\). Then up to sign, \(\det (xI - A) \mid_{x^r}\) is the sum of the \((N + 1 - r) \times (N + 1 - r)\) principal minors of \(A\). We will show that \(\det (xI - A) \mid_{x}\) must be nonzero, and hence \(\text{rank}(A) = N\). We argue as follows.

Let \(M_j\) be the \(N \times N\) principal minor of \(A\) corresponding to the \(j\)-th diagonal entry. In the expansion of \(M_j\) from first principles, we have

\[
M_j = \sum_{\sigma} (-1)^{i(\sigma)} \prod_{i=1}^{N+1} a_{i\sigma_i},
\]

in which the summation is over all permutations (in fact derangements) \(\sigma\) of the index set \(\{1, \ldots, j-1, j+1, \ldots, N+1\}\), and \((-1)^{i(\sigma)}\) is the sign of \(\sigma\). Clearly the nonzero terms in the sum in (2) are of the form \(\pm \xi^e\), for various \(e \in \{0, 1, \ldots, q-1\}\). Since \(A\) has zero diagonal and nonzero off-diagonal entries, the sum \(\sum(-1)^{i(\sigma)}\) over such terms in \(M_j\) is given by

\[
\det (J - I) = (-1)^{N-1}(N-1)
\]

where \(J\) is the \(N \times N\) matrix of 1’s and \(I\) is the \(N \times N\) identity matrix. Since this is true for every \(M_j\), we conclude that

\[
\det (xI - A) \mid_{x} = \sum_{j=1}^{N+1} M_j = c_{q-1} \xi^{q-1} + \cdots + c_1 \xi + c_0,
\]

with

\[
c_{q-1} + \cdots + c_1 + c_0 = (-1)^{N-1}(N-1)(N+1)\tag{3}
\]

Now by way of contradiction, assume that

\[
c_{q-1} \xi^{q-1} + \cdots + c_1 \xi + c_0 = 0.
\]

Setting

\[
f(t) = c_{q-1} t^{q-1} + \cdots + c_1 t + c_0,
\]

we then have \(f(\xi) = 0\). Furthermore, \(f(t)\) has integral coefficients. Therefore, the \(q\)-th cyclotomic polynomial \(\Phi_q(t)\) must divide \(f(t)\). Note also from (3) that \(f(1) \equiv (-1)^N \pmod{p}\). Writing
\( f(t) = \Phi_q(t)h(t) \), we must have that \( \Phi_q(1)h(1) \equiv (-1)^N \pmod{p} \). In particular, \( \Phi_q(1) \not\equiv 0 \pmod{p} \). But we can easily show that for \( m = p^r \) with \( r > 0 \) and \( p \) prime, we must have \( \Phi_m(1) = p \).

To see this, recall that

\[
t^m - 1 = \prod_{d \mid m} \Phi_d(t)
\]

(see, for example, [3]), and thus, by Möbius inversion,

\[
\Phi_m(t) = \prod_{d \mid m} (t^d - 1)^{\mu\left(\frac{m}{d}\right)}.
\]

In (4), \( \mu \) is the Möbius function defined by

\[
\mu(m) = \begin{cases} 
1 & \text{if } m = 1 \\
(-1)^\nu & \text{if } m \text{ is a product of } \nu \text{ distinct primes}, \\
0 & \text{otherwise}.
\end{cases}
\]

It immediately follows that for \( m = p^r, r > 0 \),

\[
\Phi_m(t) = \frac{t^{p^r} - 1}{t^{p^r-1} - 1} = 1 + t^{p^r-1} + t^{2p^r-1} + \ldots + t^{(p-1)p^{r-1}},
\]

and so \( \Phi_m(1) = 1 \). This gives us the desired contradiction.

We note that the property of \( \Phi_m(1) \) for \( m = p^r \) that we have made use of is a special case of the following more general result

\[
\Phi_m(1) = \begin{cases} 
0 & \text{iff } m = 1 \\
p & \text{iff } m = p^r, \text{ } p \text{ prime, } r > 0 \\
1 & \text{iff } m \text{ has two or more prime factors},
\end{cases}
\]

which can be found in [1].

In proving Theorem 1 we used the fact that the row sums of the matrix \( A \) vanish only to show that \( \text{rank}(A) < N + 1 \). The same argument used in the proof also provides a combinatorial proof of the following linear algebra result:

**Theorem 2** Suppose \( A \) is an \( N \times N \) zero diagonal matrix whose off-diagonal entries are \( q \)-th roots of unity for some \( q = p^r, p \) prime, \( r > 0 \). If \( N \not\equiv 1 \pmod{p} \), then \( A \) is nonsingular.
Remarks: Note that Theorem 2 and its proof apply more generally to a matrix whose diagonal entries are algebraic integers which are merely divisible by the prime p.

Furthermore, if q is not a prime power, then we can show that the conclusion of Theorem 1 is false. In this case $q = uv$ with $\gcd(u,v) = 1$. Using the Chinese remainder theorem, pick $t < q$ with $t \equiv 0 \pmod{u}$ and $t \equiv 1 \pmod{v}$. Take $z_1 = \cdots = z_t = 1$ and $z_{t+1} = \cdots = z_{qn+1} = 0$. Then the twin identities

$$1 + \xi^u + \cdots + \xi^{u(t-1)} = 0, \quad 1 + \xi^u + \cdots + \xi^{u(t-2)} = 0$$

show that no matter which $z_i$ is discarded, the remaining ones can be multiplied by $q$-th roots of 1 using $n$ copies of each root in such a way that they sum to 0.

Finally, we can consider the variant of the problem in which the classes $S_{i,0}, S_{i,1}, \ldots, S_{i,q-1}$ are not required to have the same cardinality. In this case Theorem 2 implies that the solution, if it exists, must be unique up to scalar multiples. It is easy to see that the sequence 1, 1, 1, 3, 3 for example, admits a solution in this general sense.

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References

