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A Bijective Proof for the Number of Labeled q-Trees

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ABSTRACT

We give a bijective proof that the number of vertex labeled q-trees on n vertices is given by

$$\binom{n}{q}[\,q\,n - q^2 + 1\,]^{n-q-2} \quad .$$

The bijection transforms each pair (S, f) where S is a q-element subset of an n-set, and f is a function mapping an $(n - q - 2)$-set to a $(q\,n - q^2 + 1)$-set into a labeled q-tree on n nodes by a cut-and-paste process. As a special case, $q = 1$ yields a new bijective proof of Cayley's formula for labeled trees. The general bijection also provides for the enumeration of labeled q-trees in which a specified subset of the vertices forms a clique.

1. INTRODUCTION

For each $q \geq 1$, the class of graphs called q-trees is defined recursively as follows. The smallest q-tree is the complete graph K_q on q vertices. A q-tree with $n > q$ vertices is obtained from a q-tree H on $n - 1$ vertices by introducing a new vertex v, which is connected to all the vertices of a complete graph K_q in H. For $q = 1$ this recursion defines the class of trees, and thus q-trees are generalizations of ordinary trees. We will denote by C_n^q the set of *labeled* q-trees on n nodes. Cayley [5] was the first to enumerate C_n^1:

$$|C_n^1| = n^{n-2} \quad . \qquad 1.1$$

In his honor, labeled 1-trees are sometimes referred to as Cayley trees. There are a number of analytic proofs of this famous formula in the literature, and a collection of these appears in Moon [13]. Prufer [15] was the first to provide a bijective proof of 1.1, and in 1981, Joyal constructed an elegant encoding for bi-rooted Cayley trees from which 1.1 follows [12]. More recently in 1986, a class of bijective proofs that yields Cayley's formula and the number of spanning trees of various other graphs as well, was given by

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Egecioglu and Remmel [9].

In the general case, the number of labeled q-trees on $n > q+1$ nodes is given by

$$|C_n^q| = \binom{n}{q} [\, q\, n - q^2 + 1 \,]^{\, n - q - 2} \quad . \qquad 1.2$$

As an example, there are six labeled 2-trees on 4 vertices as depicted in Figure 1. In this case there is only one underlying 2-tree.

Figure 1

Some of the approaches to the proof of Cayley's formula that appear in [13] such as the Lagrange Inversion Formula have been applied to prove 1.2 for 2-trees. Since the original proofs involve rather complicated combinatorial analyses, the computations are difficult to generalize to arbitrary dimensional case. In 1969, Beineke and Pippert [3] extended Dziobek's proof [8] of Cayley's formula to prove 1.2 for arbitrary q . This approach relies on an identity derived from one of Abel's formulas [1]. Later, using somewhat different methods, Moon in [14] generalized Clarke's inductive proof [7] for the $q = 1$ case and gave another analytic proof of 1.2 . A summary of the various approaches for the proof of 1.2 for the case $q = 2$ can be found in [2]. However, none of the known proofs for the number of labeled q-trees in the general case is bijective, even though the right hand side of 1.2 has an easy combinatorial interpretation:

Let $|K_q|$ and $|K_{q+1}|$ denote the number of complete graphs K_q and K_{q+1} contained in a q-tree on n nodes, respectively. It is easy to show by induction that $|K_q| = q\, n - q^2 + 1$ and $|K_{q+1}| = n - q$. Hence 1.2 can be written in the form

$$\binom{n}{q} |K_q|^{\,|K_{q+1}| - 2} \qquad 1.3$$

Thus the right hand side of 1.2 can be interpreted as enumerating the number of pairs (S , f) where S is a q-element subset of an n-set, and f is a function mapping an $(\, |K_{q+1}| - 2\,)$-set to a $|K_q|$-set.

In this paper we present a bijective proof of 1.3 for the number of labeled q-trees. Our method makes use of the Egecioglu-Remmel (ER) bijection for Cayley trees to construct a bijection between the pairs (S, f) and C_n^q. In our bijection, such functions f are further decomposed roughly in the following way: Each K_{q+1} which is a subgraph of a given q-tree G consists of $q+1$ "faces" which are K_q's. One of these faces will be fixed for all but one of the K_{q+1}'s in G. As $|K_q| = q\,|K_{q+1}| + 1$, f will be interpreted as dictating how these $n-q$ K_{q+1}'s should be glued together along their faces to form G. The actual gluing process is accomplished by resorting to the ER bijection for Cayley trees. It is interesting to note that even though we start with a bijection for Cayley trees as the point of departure, our proof for the general case specialized to $q=1$ yields yet another bijection for the number of Cayley trees different from the ER bijection.

As is the case with most bijective proofs, we gain extra information about the underlying combinatorial objects by considering the special properties of the bijection. In this instance, we obtain enumerative results of the following kind:

Suppose $\{i, j, k\} \subset \{1, 2, \ldots, n\}$. *Then*

(i) *the number of 2-trees on n nodes in which the vertices i and j are adjacent is*

$$\left[\binom{n}{2} - \binom{n-2}{2}\right][2n-3]^{n-4} = (2n-3)^{n-3} ,$$

(ii) *the number of 2-trees on n nodes in which the vertices i, j, and k form a triangle is*

$$3(2n-3)^{n-4} .$$

(iii) *the number of Cayley trees on n nodes in which the vertices i and j are adjacent is*

$$2n^{n-3} . \qquad 3.5$$

We give the general form of the above formulas for arbitrary q and a subset of m labels (Corollary 4.1) as well:

The number of labeled q-trees on $n > q+1$ nodes in which the vertices $v_1, v_2, \ldots, v_m \in \{1, 2, \ldots, n\}$ form a clique is

$$\left[\sum_{i=0}^{q+1-m} (q+1-i)\binom{n+i-q-2}{i}\right] \times (nq - q^2 + 1)^{n-q-2} .$$

The outline of this paper is as follows. In section 2 we reproduce the ER bijection (the θ_n bijection in [9]) for the number of Cayley trees and present requisite terminology concerning q-trees. In section 3, we construct our bijection for the cases of labeled 2-trees and Cayley trees. The description of the bijection for arbitrary q is presented in Section 4.

2. PRELIMINARIES

2.1 ER bijection for Cayley trees

For completeness we reproduce here the θ_n bijection for Cayley trees that appears in [9].

Denote by C_n the set of Cayley trees on n nodes rooted at the largest labeled node n. Furthermore, we orient each edge $\{ i , j \}$ of a Cayley tree in C_n by directing it toward the root. Clearly, $|C_n| = |C_n^1|$. Next, let F_n denote the set of functions from $\{ 2 , \ldots , n-1 \}$ into $\{ 1 , \ldots , n \}$. The bijection θ_n between C_n and F_n is most easily described by referring to an explicit example.

Suppose $n = 21$ and $f \in F_{21}$ is given by the following table

i	$f(i)$	i	$f(i)$	i	$f(i)$	i	$f(i)$
2	5	7	7	12	20	17	16
3	4	8	12	13	19	18	6
4	5	9	1	14	19	19	7
5	3	10	4	15	6	20	12
6	21	11	4	16	1		

We can view f as a directed graph with vertex set $\{ 1 , \ldots , 21 \}$ by putting an edge from i to j if $f(i) = j$. For example, the digraph for f given above is pictured in Figure 2.

Figure 2

A moment's thought will convince one that in general, the digraph corresponding to an $f : \{ 2 , \ldots , n-1 \} \to \{ 1 , \ldots , n \}$ will consist of two trees rooted at 1 and n, respectively, with all edges directed toward their roots plus a number of directed cycles of length ≥ 1 where for each vertex v on a given cycle, there is possibly a tree attached to v with v as the root and all edges directed toward v. Note that there are trees rooted at 1 and n due to the fact that 1 and n are not in the domain of f, so that there are no directed edges out of 1 or n. Note also that cycles of length one or loops simply correspond to fixed points of f.

As in Figure 2, we imagine the directed graph corresponding to $f \in F_n$ is drawn so that

(a) the trees rooted at 1 and n are drawn on the extreme left and extreme right respectively with their edges directed upwards,

(b) the cycles are drawn so that their vertices form a directed path on the line between 1 and n with one backedge above the line and the tree attached to any vertex on a cycle is drawn below the line between 1 and n with edges directed upwards,

(c) each cycle is arranged so that its smallest element is on the right and the cycles themselves are ordered from left to right by increasing smallest elements.

Once the directed graph for f is drawn as above, let us refer to the rightmost element in the i th cycle as r_i and the leftmost element in the i th cycle as l_i. Thus for the f given above, $l_1 = 4$, $r_1 = 3$, $l_2 = r_2 = 7$, $l_3 = 20$, and $r_3 = 12$. Once an $f \in F_n$ is drawn in this manner, it is easy to describe the bijection $\theta_n(f)$. That is, if the directed graph of f has k cycles where $k > 0$, we simply eliminate the backedges $r_i \to l_i$ for $i = 1, \ldots, k$ and add the edges $1 \to l_1$, $r_1 \to l_2$, $r_2 \to l_3, \ldots, r_k \to n$. For example, in Figure 2, we eliminate the backedges $3 \to 4$, $7 \to 7$, $12 \to 20$ and add the edges $1 \to 4$, $3 \to 7$, $7 \to 20$, and $12 \to 21$ which are dotted for emphasis. If there are no cycles in the directed graph of f, i.e., $k = 0$, then we simply add the edge $1 \to n$.

Note that it is immediate that θ_n is a bijection between F_n and C_n since given any Cayley tree $T \in C_n$, we can easily recover the directed graph of $f \in F_n$ such that $\theta_n(f) = T$. The key point here is that by our conventions for the ordering of the cycles of f, it is easy to recover the sequence of nodes $r_1, r_2, \ldots, r_k$ since r_1 is the smallest element on the path between 1 and n, r_2 is the smallest element on the path between r_1 and n, etc., and clearly, knowing $r_1, r_2, \ldots, r_k$ allows us to recover f from T.

Now we have that $\theta_n : F_n \to C_n$ is a bijection so that we arrive at Cayley's formula $n^{n-2} = |F_n| = |C_n^1|$.

2.2 Definitions and some properties of q-trees

The following definitions which are analogues of the notions of an ordinary path, walk, etc. in a simple graph G are similar to but slightly different from those introduced by Beineke and Pippert [4], Harary and Palmer [11] and Winkler [16]. We will consider sequences of K_q's and K_{q+1}'s to go from one K_{q+1} subgraph of a q-tree to another K_{q+1} subgraph. More precisely, we define a $q-walk$ π in a simple graph as an alternating sequence of K_{q+1}'s and K_q's

$$\pi = (\sigma_0 , \rho_1 , \sigma_1 , \ldots , \sigma_{n-1} , \rho_n , \sigma_n)$$

starting and ending with K_{q+1}'s σ_0 and σ_n, such that each K_{q+1} graph σ_i contains ρ_{i-1} and ρ_i as distinct K_q's. n is the length of π. A *$q-path$* is a q-walk where all the graphs in the sequence are distinct. In this case we say that the two K_{q+1}'s σ_0 and σ_n are joined by a q-path. Figure 3 depicts a 2-tree on 12 nodes. Denoting edges and triangles by the labels of their respective vertices, the sequence $(679, 69, 369, 36, 356, 35, 358)$ is a 2-path of length three joining the triangles 679 and 358.

Figure 3

A q-walk is a *$q-circuit$* if its length is at least three, $\sigma_0 = \sigma_n$, and all other elements in the sequence are distinct.

We shall make use of the following property of q-trees.

If G is a q-tree, then every pair of K_{q+1}'s in G are joined by a unique q-path. In particular G contains no q-circuits.

The proof of this follows from a similar result of Beineke and Pippert [4] and will be omitted.

Next, we need the notion of a q-path connecting two vertices instead of two K_{q+1}'s in a q-tree. In order to distinguish this type of a path between vertices from a q-path which joins two K_{q+1}'s, we will refer to them as *$q-trails$*.

Definition *Suppose v_i and v_j are two nonadjacent vertices of a q-tree. A q-trail between v_i and v_j is a q-path $\pi = (\sigma_i , \rho_1 , \sigma_1 , \ldots , \sigma_{k-1} , \rho_k , \sigma_j)$ such that v_i belongs to σ_i, v_j belongs to σ_j, and neither v_i nor v_j belongs to any other graph in the sequence.*

For example , the 2-path depicted in Figure 3 constitutes a 2-trail between the vertices 7 and 8 . Since a vertex may belong to more than one K_{q+1}, it is not evident from the definition above that a q-trail between two vertices is unique. We next prove

Lemma 2.1 *Suppose v_i and v_j are two nonadjacent vertices of a q-tree G. Then there is a unique q-trail between v_i and v_j.*

Proof

We proceed by contradiction. Suppose there are two distinct q-trails

$$\pi^1 = (\sigma_i^1 , \rho_1^1 , \sigma_1^1 , \ldots , \sigma_{k-1}^1 , \rho_k^1 , \sigma_j^1)$$
$$\pi^2 = (\sigma_i^2 , \rho_1^2 , \sigma_1^2 , \ldots , \sigma_{l-1}^2 , \rho_l^2 , \sigma_j^2)$$

between v_i and v_j . Since there is a unique q-path between any pair of K_{q+1} 's in a q-tree, we have that $\sigma_i^1 \neq \sigma_i^2$ or $\sigma_j^1 \neq \sigma_j^2$. Let $N_G(v_i)$ and $N_G(v_j)$ denote the subgraphs induced by the neighborhoods of v_i and v_j respectively. Let H_i be the subgraph induced by $N_G(v_i) \cup v_i$ and let H_j be the subgraph induced by $N_G(v_j) \cup v_j$. Then both $N_G(v_i)$ and $N_G(v_j)$ are $(q-1)$-trees [10], and therefore by Corollary 3 of [6], the subgraphs H_i and H_j are themselves q-trees. As such, there is q-path τ^i between the K_{q+1} 's σ_i^1 and σ_i^2 that lies entirely in H_i . Similarly, there is q-path τ^j between the K_{q+1} 's σ_j^1 and σ_j^2 that lies entirely in H_j . Since v_i and v_j are not adjacent, the sequence $(\tau^i , \pi^2 , \tau^j , \pi^1)$ (where τ^2 and π^1 are traversed in the opposite direction), forms a q-circuit in G , which is a contradiction. •.

Note that the idea of a q-trail between two vertices v_i and v_j of a q-tree can be modified slightly to define the notion of a q-path joining a vertex v_i and a K_{q+1} subgraph σ_k . Here we require that v_i belong only to the first $K_{q+1} = \sigma_i$ in the q-path

$$(\sigma_i , \rho_1 , \sigma_1 , \ldots , \sigma_{k-1} , \rho_k , \sigma_k)$$

joining σ_i and σ_k . From our arguments above, it is not difficult to see that there exists a unique q-path joining a vertex v_i and any K_{q+1} that does not contain v_i . As an example, the sequence $(679 , 69 , 369 , 36 , 356 , 35 , 358)$ is the 2-path in the 2-tree pictured in Figure 3 which joins vertex 7 to triangle 358 .

In analogy with the notion of external nodes of an ordinary tree, a *leaf* in a q-tree G is defined as a vertex of degree q . It is not difficult to prove that any q-tree G on $n \geq q+1$ vertices contains at least two leaves [6] , [10]. From this fact it follows that any clique K in a q-tree T with $n > q$ nodes is contained in at least one clique of size $q+1$.

An *acyclic orientation* of a simple loopless graph G is an assignment of a direction to each of the edges of G in such a way that no directed cycles are formed. By abuse of language, we will call a complete graph K_{q+1} simply *oriented* if it is equipped with an acyclic orientation. Clearly, this is equivalent to putting a total order on the vertices of K_{q+1} . We let the unique vertex with no outgoing edges (i.e. a *sink* in digraph terminology) correspond to the smallest element in this total order. Given an oriented K_{q+1} , this total order defines $q+1$ *faces* where the i th face is the oriented K_q subgraph of K_{q+1} induced by all the vertices except the i th smallest one. Given an oriented K_{q+1} , its *largest* face will be referred to as the *top* or the *top face*. In other words, the top face of an oriented K_{q+1} is the K_q induced by all the vertices minus the unique *source* (i.e. no incoming edges) vertex in K_{q+1} . Figure 4 illustrates these notions for $q+1 = 4$.

Figure 4

As an extension of the above definition, a q-tree with an acyclic orientation will be referred to as an *oriented* q-tree. It will be useful to single out the following types of oriented q-trees for our purposes: An oriented q-tree will be referred to as *well-oriented q-tree* if it belongs to the class of graphs defined by the following recursive definition

i) The smallest well-oriented q-tree is an oriented K_{q+1} ,

ii) An oriented q-tree on $n > q + 1$ vertices is well-oriented if all of its leaves are sources and there exists a leaf vertex whose removal results in a well-oriented q-tree.

Note that there is exactly one well-oriented q-tree on $q + 2$ nodes. It is obtained by attaching a source to all the vertices in the top face of K_{q+1} .

Figure 5

3. BIJECTIONS FOR 2-TREES AND CAYLEY TREES

3.1 The 2-tree bijection

In this section we describe our bijection Θ_n^2 for 1.3 for 2-trees. This bijection is constructed in two phases. First we give a combinatorial interpretation to functions f mapping an $(n-4)$-set to a $(2n-3)$-set in terms of triangle labeled q-trees. Then we show how to obtain labeled 2-trees from this class if we are given a 2-subset of an n-set. We shall call the first intermediate bijection Ω_n^2 and the second one Γ_n^2 . Thus the actual bijection Θ_n^2 will be the composition $\Gamma_n^2 \circ \Omega_n^2$.

Note that for the case $q = 2$, K_q 's become edges and K_{q+1} 's become triangles. Suppose we are considering 2-trees on $n > 3$ nodes. As we have remarked before, the right hand side of 1.3 can now be written in the form

$$\binom{n}{2}(2\,|K_3| + 1)^{|K_3| - 2} = \binom{n}{2}[\,2\,(n-2) + 1\,]^{n-4} \qquad 3.1$$

Let F_n^2 denote the collection of all functions from the set $\{2, 3, \ldots, n-3\}$ into the set of pairs $\{1, 2, \ldots, n-2\} \times \{1, 2\} \cup \{(n-2, 3)\}$. Clearly,

$$|F_n^2| = [\,2\,(n-2) + 1\,]^{n-4} \quad . \qquad 3.2$$

We shall interpret the numerals in the domain of such a function f as oriented triangles $2, 3, \ldots, n-3$ contained in the collection pictured in Figure 6.

Figure 6

Note that in this case, the faces of the i th triangle have the following order

Figure 7

Our first task will be to construct from f a well-oriented 2-tree which we shall denote by $\Omega_n^2(f)$, in which each triangle is labeled instead of the vertices. To this end, we first construct the *weighted* functional digraph of $f \in F_n^2$ on the triangles given in Figure 6 as vertices, in the following manner: if $f(i) = (j, k)$, then we put an edge from the i th triangle to the j th triangle with weight k. As an example, suppose $n = 22$ and f is given by the following table

i	$f(i)$	i	$f(i)$	i	$f(i)$	i	$f(i)$
2	$(5,1)$	7	$(12,2)$	12	$(20,1)$	17	$(16,2)$
3	$(4,2)$	8	$(12,1)$	13	$(19,1)$	18	$(6,1)$
4	$(5,2)$	9	$(1,2)$	14	$(19,2)$	19	$(20,3)$
5	$(3,1)$	10	$(4,2)$	15	$(6,2)$		
6	$(7,1)$	11	$(4,1)$	16	$(1,1)$		

The corresponding weighted functional digraph is pictured in Figure 8.

Figure 8

Similar to the construction involved in the ER bijection, the weighted functional digraph of an $f \in F_n^2$ will consist of two trees rooted at the triangle 1 and the triangle $n-2$, respectively, with all edges directed towards their roots plus a number of directed cycles where for each vertex v on a given cycle, there may be a tree attached to v with v as the root and the edges directed toward v. Following the ER bijection, we imagine that the weighted functional digraph of f is drawn as in Figure 8 so that

(a) the trees rooted at 1 and $n-2$ are drawn on the extreme left and extreme right respectively with their edges directed upwards,

(b) the cycles are drawn so that their vertices form a directed path on the line between 1 and $n-2$ with one backedge above the line and the tree attached to any vertex on a cycle is drawn below the line between 1 and $n-2$ with edges directed upwards,

(c) each cycle is arranged so that its smallest element is on the right and the cycles themselves are ordered from left to right by increasing smallest elements.

Once the weighted digraph of $f \in F_n^2$ is drawn as above, let us refer to the rightmost element in the i th cycle as r_i and the leftmost element in the i th cycle as l_i. Thus for the f given above, we have only one cycle and thus $l_1 = 4$, $r_1 = 3$. Now if the weighted functional digraph of f has k cycles where $k > 0$, we simply eliminate the backedges $r_i \to l_i$ for $i = 1, \ldots, k$ and add the edges $1 \to l_1$, $r_1 \to l_2$, $r_2 \to l_3, \ldots, r_k \to n-2$. The weight of a backedge (r_i, l_i) is assigned to the newly added edge *preceding* it. The last edge added between $r_k \to n-2$ always has weight 3. For example, in Figure 8, we eliminate the backedge $3 \to 4$, and add the edges $1 \to 4$ and $3 \to 20$, with weights 2 and 3, respectively. If there are no cycles in the weighted functional digraph of f, i.e., $k = 0$, then we simply add the edge $1 \to n-2$ with weight 3.

Now the weights on the edges are made use of as follows: if there is a directed edge between the triangles i and j with weight k, then we identify the top face of the i th triangle with the k th face of the j th triangle. For the example above, this identification phase results in the oriented, triangle labeled 2-tree $\Omega_{22}^2(f)$ pictured in Figure 9.

Figure 9

By our construction, it is not difficult to see that in general, the resulting oriented 2-tree $\Omega_n^2(f)$ is well-oriented. However, the triangles labeled 1 and $n-2$ are not arbitrary. In fact, again by our identification process, in the 2-path

$$\pi = (1, \rho_1, \sigma_1, \ldots, \sigma_{k-1}, \rho_k, n-2)$$

joining the triangles 1 and $n-2$ in $\Omega_n^2(f)$, the edge ρ_1 is the top face of triangle 1 and the edge ρ_k is the top face of triangle $n-2$. The edge ρ_k in this path π will play a special role later. We will refer to it as the *base* of $\Omega_n^2(f)$. Let us denote all well-oriented, triangle labeled 2-trees on n nodes in which the 2-path between the triangles 1 and $n-2$ has the property above by W_n^2. Given a $T \in W_n^2$, we can easily recover the function $f \in F_n^2$ such that $\Omega_n^2(f) = T$. By our conventions of ordering the cycles of f, it is easy to recover the sequence of vertices $r_1, r_2, \ldots, r_k$ which are the smallest elements on the directed cycles of f as in the ER bijection. This is because r_1 is the smallest element on the 2-path between triangle 1 and triangle $n-2$, r_2 is the smallest element on the 2-path between r_1 and $n-2$, etc., and knowing $r_1, r_2, \ldots, r_k$ allows us to recover the structure of the functional digraph of f. The weights on the edges can then be read off from the way the identifications are made since the tail of a directed edge (i, j) always emanates from the top face of triangle i. Instead of reworking the above example we gave backwards, we present in Figure 10, $[2(5-2)+1]^{5-4} = 7$ elements of W_5^2 and the corresponding functions $f_1, \ldots, f_7$ in F_n^2 under the bijection Ω_5^2, along with their weighted functional digraphs.

Figure 10

In the light of the construction described above , we have

Lemma 3.1 Ω_n^2 *is a bijection between* F_n^2 *and* W_n^2 .

Now let S_n^2 denote the collection of 2-subsets of $\{ 1 , 2 , \ldots , n \}$. Our next task is to construct a bijection Γ_n^2 between the space $S_n^2 \times W_n^2$ and labeled 2-trees on n nodes, C_n^2 . The definition of Γ_n^2 depends on whether or not a given subset $S \in S_n^2$ contains the special elements 1 or $n - 2$. We first consider the straightforward case in which neither 1 nor $n - 2$ is in $S = \{ i , j \}$:

Case I : $S \cap \{ 1 , n - 2 \} = \emptyset$

Recall that the path between triangle 1 and triangle $n - 2$ meets the latter at the base ρ_k (which actually is the top face of triangle $n - 2$ since we are considering elements of W_n^2). Now we assign the elements $i < j$ of S to the vertices of the base, according to the linear order that is dictated. In other words, the smaller of the two elements i is assigned to the sink on ρ_k and the larger element j is assigned to the source. After this, we first change the label of the i th triangle to $n - 1$, and the label of the j th triangle to n . Next, we "push" the labels on the triangles away from the edge ρ_k to their source vertices. For example, the source vertex of the $n - 2$ triangle is labeled $n - 2$, then the vertices of the triangles sharing an edge with the $n - 2$ triangle are labeled, etc. Note that this process can be carried out because of the way that the elements of W_n^2 have been constructed. After all the vertices have been labeled, we may disregard the orientation on the edges. Using the previous example $\Omega_{22}^2 (f)$ we have constructed (Figure 9) together with the subset $\{ 3 , 12 \} \subset S_{22}^2$, this process results in the labeled 2-tree depicted in Figure 11:

Figure 11

Note that in Case 1, the vertices labeled 1 and $n-2$ in the resulting labeled 2-tree are never adjacent. If $S \cap \{1, n-2\} \neq \emptyset$, then the labeling process that we shall describe momentarily will make sure that these two labels will be assigned to adjacent vertices. We need only to change the relabeling process for the triangles. The labeling of the vertices themselves will then be carried out as in Case 1.

Case II : $S \cap \{1, n-2\} \neq \emptyset$

This case is split into three subcases depending on whether or not $S \cap \{1, n-2\} = \{1\}, \{n-2\}$, or $\{1, n-2\}$.

Subcase II A : $S = \{1, j\}, j \neq n-2$

As in Case I, assign the elements of S as labels to the vertices of the base. Relabel triangle j as $n-1$, triangle 1 as $t = \max\{\{n-1, n\} - \{j\}\}$. Thus triangle 1 is relabeled $n-1$ or n depending on whether or not $j = n$ or $j < n$, respectively. Now push the triangle labels to their respective source vertices.

Subcase II B : $S = \{i, n-2\}, i \neq 1$

Assign the elements of S as labels to the vertices of the base. Relabel triangle i as $n-1$, triangle 1 as $t = \max\{\{n-1, n\} - \{i\}\}$, and triangle $n-2$ as 1. Now push the triangle labels to their respective source vertices.

Subcase II C : $S = \{1, n-2\}$

Assign the elements of S as labels to the vertices of the base. Relabel triangle 1 as $n-1$, triangle $n-2$ by n. Push the triangle labels to their respective source vertices.

The effect of these relabelings arising from Case II on the two special triangles 1 and $n-2$ (omitting the relabelings of the rest of the triangles) before and after the labels are assigned is described graphically in Figure 12.

Figure 12

We shall now show that the correspondence $\Gamma_n^2 : S_n^2 \times W_n^2 \to C_n^2$ described above is reversible.

Lemma 3.2 Γ_n^2 *is a bijection between* $S_n^2 \times W_n^2$ *and* C_n^2 *.*

Proof

Consider a labeled 2-tree $T \in C_n^2$. From T we shall construct a pair $(S, T') \in S_n^2 \times W_n^2$ such that $\Gamma_n^2(S, T') = T$. The construction is carried out according to whether or not the vertices labeled 1 and $n-2$ in T are adjacent. As we have remarked before, the first case here will correspond to Case I of the construction of Γ_n^2 .

Case 1 : *1 and* $n-2$ *are not adjacent in* T

We look at the 2-trail π joining vertex 1 to vertex $n-2$. Suppose ρ_k is the last edge on π (this will be our base edge) and $i < j$ are the labels on the endpoints of ρ_k . Then we perform the following operations

(i) orient ρ_k from j to i ,

(ii) "bump" these two elements out and put $S = \{ i , j \}$,

(iii) reverse the process of "pushing" by sliding the labels of the vertices incident to the endpoints of edges that have already been oriented to the associated triangles while making these vertices sources,

(iv) relabel triangle $n-1$ as i and triangle n as j .

If T' is the oriented, triangle labeled 2-tree that results in after the steps (i) - (iv) have been carried out, then it is not difficult to see that $T' \in W_n^2$ and $\Gamma_n^2(S, T') = T$.

The second case we have to consider is when 1 and $n-2$ are adjacent vertices in T . We will show that the pair (S, T') can be unambiguously reconstructed from T . Since the orientation and sliding the labels to the triangles follows the same pattern as Case 1 above, we shall only indicate the relabeling of the triangles (and the extraction of the set S for clarity). Actually, in the following case, only step (iv) above will be different.

Case 2 : *1 and* $n-2$ *are adjacent in* T

The subcases here are as follows:

Subcase 2 A : $\{ 1 , n-2 , n \}$ *is a triangle in* T

Consider the 2-path from the vertex labeled $n-1$ to the triangle formed by $\{ 1 , n-2 , n \}$. By the correspondences in Figure 12, the labels of the vertices on the last edge ρ_k on this 2-path determine unambiguously one of the three Subcases II A, II B or II C in the construction of Γ_n^2 . As an example, if the labels on the endpoints of ρ_k are

$\{ n-2 , n \}$, then we see that we have to reverse the transformation in Subcase II B. To do this, we first apply steps (i) - (iii) as in Case 1 above. This gives an orientation to T and yields $S = \{ n-2 , n \}$. But in the application of step (iv) we relabel triangle 1 as $n-2$, and triangle $n-1$ as 1 .

Subcase 2 B : $\{ 1 , n-2 , n \}$ *is not a triangle in* T

Suppose at first that 1 and n are not adjacent. If one of the labels on the last edge ρ_k on the 2-trail from n to 1 is $n-2$, then we reverse the steps according to Subcase II B. Thus $S = \{ i , n-2 \}$ where i is the other label on ρ_k . In step (iv) then, we relabel triangle $n-1$ as i , triangle 1 as $n-2$, and triangle n as 1 . If $n-2$ is not a label on ρ_k , then we are in Subcase II A, but to find the base edge we consider the 2-trail from n to $n-2$. Note that this is well-defined since now n and $n-2$ cannot be adjacent. In this case, $S = \{ 1 , j \}$ (where j is the label of the other vertex on the base) and step (iv) results in the relabelings of the triangles $n-1$ by j and n by 1 .

If 1 and n are adjacent, then $n-2$ and n cannot be adjacent, and in this case we reverse the transformation in Subcase II A. •

As a nontrivial example, Figure 13 depicts the pair $(\Gamma_{12}^2)^{-1} (T)$ where T is the 2-tree on 12 nodes given in Figure 3 .

Figure 13

Combining Lemma 3.1 and Lemma 3.2, we have

Theorem 3.1 *The map* $\Theta_n^2 = \Omega_n^2 \circ \Gamma_n^2$ *is a bijection between* $S_n^2 \times F_n^2$ *and* C_n^2 *. Thus*

$$| C_n^2 | = \binom{n}{2} [2n-3]^{n-4} \quad .$$

Note that by the construction of the bijection Θ_n^2 , the vertices labeled 1 and $n-2$ are adjacent in a 2-tree on n nodes if and only if $S \cap \{ 1 , n-2 \} \neq \emptyset$. Similarly, the vertices $1 , n-2 , n$ form a triangle in T exactly in the cases where $S = \{ 1 , n \}$ (Subcase II A),

$S = \{ n-2 , n \}$ (Subcase II B), and $S = \{ 1 , n-2 \}$ (Subcase II C). Since we could have used any three labels $\{ i , j , k \} \subset \{ 1 , 2 , \ldots , n \}$ to play the role of 1, $n-2$, and n in the construction of Θ_n^2, we immediately have

Corollary 3.1 *Suppose* $\{ i , j , k \} \subset \{ 1 , 2 , \ldots , n \}$. *Then*

i) *the number of 2-trees on n nodes in which the vertices i and j are adjacent is*

$$\left[\binom{n}{2} - \binom{n-2}{2} \right] [2n-3]^{n-4} = (2n-3)^{n-3} \quad , \qquad 3.3$$

ii) *the number of 2-trees on n nodes in which the vertices i, j, and k form a triangle is*

$$3 (2n-3)^{n-4} \quad . \qquad 3.4$$

3.2 The Cayley tree bijection

After defining the bijection Θ_n^2 for labeled 2-trees, we now show how to use a similar construction to arrive at a bijective proof of 1.1. Note that we are forced to interpret 1.1 in the form

$$| C_n^1 | = n\, n^{n-3} \quad ,$$

where the first factor $n = \binom{n}{1}$ corresponds to the selection of an element from the set $\{ 1 , 2 , \ldots , n \}$, and the second factor enumerates W_n^1. Clearly, the bijection Θ_n^1 will be rather more complicated than the ER bijection, since it uses the ER bijection as a subprocedure. As we shall indicate in the next section, both of these constructions for $q = 2$ and $q = 1$ are special cases of the bijection Θ_n^q for q-trees, but the elegant bijective proofs that exist for the proof of 1.1 do not seem to generalize to arbitrary q. Hence even though bijection Θ_n^1 is natural in the setting of q-trees, it probably is not the easiest bijection if we are only interested in Cayley trees.

Let now F_n^1 denote the collection of all functions from the set $\{ 2 , 3 , \ldots , n-2 \}$ into the set of pairs $\{ 1 , 2 , \ldots , n-1 \} \times \{ 1 \} \cup \{ (n-1 , 2) \}$. Clearly,

$$| F_n^1 | = n^{n-3} \quad .$$

Since the construction of Θ_n^1 is quite similar to the construction of Θ_n^2, we shall simply outline the bijections Ω_n^1 and Γ_n^1 without going into too much detail, and illustrate these bijections on an example. First, however, we interpret the numerals $2 , 3 , \ldots , n-2$ in the domain of F_n^1 as oriented edges as in Figure 14, since in this case K_{q+1}'s are simply edges.

Figure 14

Here the faces of the i th edge have the following order

Figure 15

The weighted functional digraph of an $f \in F_n^1$ has the oriented edges in Figure 14 as its vertex set. We again put an arc from edge numbered i to edge numbered j with weight k if $f(i) = (j , k)$. In this case the only arcs with weight 2 point to the edge $n-1$ and all the other weights are 1 . We draw the weighted functional digraph of f as in the ER bijection. Thus the tree with root 1 appears at the extreme left, the one with root $n-1$ appears at the extreme right, and the remaining components are drawn with the smallest element on each cycle on the right. The cycles themselves are ordered with increasing smallest elements. Breaking the backedges, adding the necessary edges between the cycles as in the ER bijection, and then performing the identifications results in a well-oriented, edge labeled 1-tree $T' = \Omega_n^1 (f)$.

The construction of Γ_n^1 is similar to the construction of Γ_n^2 . Given a pair $(S , T') \in S_n^1 \times W_n^1$, with $S = \{ i \}$, we distinguish between the special cases where $i \neq 1 , n-1$, $i = 1$, and $i = n-1$.

Case a : $i \neq 1 , n-1$

In this case, we consider the path from edge 1 to edge $n-1$. Here the base is simply the top of the edge $n-1$. At this point this vertex is assigned the label i , and the edge i is relabeled as n . Then we push the labels on the edges to the source vertices on the edges. Now discarding the orientations on the arcs results in a Cayley tree T . Note that in this case the vertices 1 and $n-1$ can never be adjacent in T .

Case b : $i = 1$

We proceed as in Case a up to the point where the edges are relabeled. Here we relabel edge 1 as n . Note that in the resulting Cayley tree T , vertices 1 and $n-1$ are adjacent and 1 lies on the path from n to $n-1$.

Case c : $i = n-1$

Again we proceed as in Case a up to the point where the edges are relabeled. In this case, edge 1 is relabeled as n , and edge $n-1$ is relabeled as 1 . Note that in the resulting Cayley tree T , 1 and $n-1$ are adjacent and that vertex $n-1$ lies on the path from n to 1 .

The three cases above define the correspondence Γ_n^1 . Because of the distinguishing features of each of these cases, it is not difficult to verify that Γ_n^1 is a bijection. Thus $\Theta_n^1 = \Omega_n^1 \circ \Gamma_n^1$ is a bijection between $S_n^1 \times F_n^1$ and Cayley trees and we have arrived at Cayley's formula in an alternate way.

We illustrate the above construction on an example. Consider the following function $f \in F_{21}^1$.

i	$f(i)$	i	$f(i)$	i	$f(i)$	i	$f(i)$
2	(5, 1)	7	(12, 1)	12	(20, 1)	17	(16, 1)
3	(4, 1)	8	(12, 1)	13	(19, 1)	18	(6, 1)
4	(5, 1)	9	(1, 1)	14	(19, 1)	19	(20, 2)
5	(3, 1)	10	(4, 1)	15	(6, 1)		
6	(7, 1)	11	(4, 1)	16	(1, 1)		

The corresponding weighted functional digraph of f and $\Omega_{21}^1(f) \in W_{21}^1$ are pictured in Figure 16. For $S = \{7\}$, the resulting Cayley tree $\Theta_{21}^1(\{7\}, f)$ is identical to the Cayley tree depicted in Figure 2.

Figure 16

We leave it to the reader to verify that if we take $S = \{ 20 \}$ (Case c above), then $\Theta^1_{21} (\{ 20 \} , f)$ is the Cayley tree in Figure 17.

Figure 17

Note that the argument given for the proof of Corollary 3.1 is valid for the bijection Θ^1_n as well. In other words, only when $i = 1$ (Case b), and $i = n - 1$ (Case c) the labels 1 and $n - 1$ turn out to be adjacent in the resulting tree. Since any two labels i and j can play the role of 1 and $n - 1$ in the definition of Θ^1_n , we have the following version of Corollary 3.1 for Cayley trees:

Corollary 3.2 *Suppose* $\{ i , j \} \subset \{ 1 , 2 , \ldots , n \}$. *Then the number of Cayley trees on* n *nodes in which the vertices* i *and* j *are adjacent is*

$$2 n^{n-3} \quad . \tag{3.5}$$

Note that by the ER bijection, 3.5 also counts the number of cycle-free functions mapping $\{ 2 , 3 , \ldots , n - 1 \}$ into $\{ 1 , 2 , \ldots , n \}$.

4. THE q-TREE BIJECTION

Now we are in a position to describe our bijection Θ^q_n for arbitrary q . Let F^q_n denote the collection of all functions f mapping $\{ 2 , 3 , \ldots , n - q - 1 \}$ into the set of pairs $\{ 1 , 2 , \ldots , n - q \} \times \{ 1 , 2 , \ldots , q \} \cup \{ (n - q , q + 1) \}$. We have

$$| F^q_n | = [q (n - q) + 1]^{n-q-2} \quad .$$

Let S_n^q and W_n^q denote the family of q-subsets of the set $\{1, 2, \ldots, n\}$ and the collection of well-oriented, K_{q+1} labeled q-trees on n nodes, respectively. As in the special cases we have considered before, Θ_n^q will be constructed as the composition of two bijections Ω_n^q and Γ_n^q :

$$\Theta_n^q : S_n^q \times F_n^q \xrightarrow{\quad\Omega_n^q\quad} S_n^q \times W_n^q \xrightarrow{\quad\Gamma_n^q\quad} C_n^q$$

where $\Omega_n^q : F_n^q \to W_n^q$.

The construction of the bijection Ω_n^q is similar to the $q = 2$ case. Thus, we first interpret the numerals in the domain of $f \in F_n^q$ as contained in $n - q$ oriented K_{q+1}'s $\sigma_1, \sigma_2, \ldots, \sigma_{n-q}$. Using these σ_i's as the vertex set, the weighted functional digraph of f is constructed by putting an edge between σ_i and σ_j with weight k if $f(i) = (j, k)$. The resulting digraph is then drawn in the manner of the ER bijection. In this way a tree with root 1 appears at the extreme left, the one with root $n - q$ appears at the extreme right, and the remaining components are drawn with the smallest element r_i on each cycle on the right. The cycles themselves are ordered with increasing order of the r_i's. After the backedges are broken and the necessary edges between the cycles added exactly as in the case of the construction of Ω_n^2 bijection, the identification of faces are made as dictated by the weights. This results in an element $T' \in W_n^q$.

Conversely, given a $T' \in W_n^q$, we consider the q-path from σ_1 to σ_{n-q} in T' . The last K_q on this q-path, say ρ_k is then the base and this gives us the top face of σ_{n-q} . Furthermore, from the σ_i's on this path, it is easy to recover the sequence $r_1, r_2, \ldots, r_k$ which will form the smallest elements on the cycles of f . Also note that the orientation of T' produces the weight between σ_i and σ_j sharing a common face. Hence we can recover $f \in F_n^q$ such that $\Omega_n^q(f) = T'$. Thus Ω_n^q is a bijection.

The construction of the Γ_n^q bijection also follows closely the $q = 2$ case. Suppose $(S, T') \in S_n^q \times W_n^q$ is given. Put $A = \{n - q + 1, n - q + 2, \ldots, n\}$. Note that the elements of A are exactly those among $\{1, 2, \ldots, n\}$ that do not appear as labels of K_{q+1}'s in T' .

Consider the q-path joining σ_1 to σ_{n-q} in T' . Recall that ρ_k , which is the last K_q on this q-path is the base, and it forms the top face of σ_{n-q} . At this point, the construction of $\Gamma_n^q(S, T')$ will consist of roughly the following steps:

(1) Assign the labels in the set S to the vertices of the base, according to the linear order dictated by its orientation.

(2) Now possibly some of the labels that appear on the base also appear as labels of K_{q+1}'s in T' . Thus we need to incorporate the missing labels $A - S$ by relabeling the K_{q+1}'s whose labels already appear in S . Having done this, we push the labels on the K_{q+1}'s to their source vertices. Note that this process is well-defined because of the way elements of W_n^q have been constructed. The resulting labeled q-tree (after the orientation

is ignored) is $T = \Gamma_n^q (S , T')$.

The difficult part in the above procedure is the relabeling phase in (2) above. We have to make sure that it will be possible to distinguish the base of a given $T \in C_n^q$ in order to recover unambiguously the corresponding pair (S , T') that gives rise to T .

As in the special cases $q = 1$ and $q = 2$, we need to consider the situations in which $S \cap \{ 1 , n - q \} = \emptyset$ and $S \cap \{ 1 , n - q \} \neq \emptyset$. In each case, we shall indicate how the relabeling is to be carried out. The following notation will be convenient for our description : By an assignment symbol of the form $X := Y$ where $X , Y \subset \{ 1 , 2 , \ldots , n \}$ and $|X| = |Y|$, we shall mean that all of the σ_i 's for $i \in X$ are to be relabeled by the elements of the set Y , following the original order. In other words, if $X = \{ x_1 , x_2 , \ldots , x_k \}$ and $Y = \{ y_1 , y_2 , \ldots , y_k \}$ with $x_1 < x_2 < \cdots < x_k$, $y_1 < y_2 < \cdots < y_k$ then after the relabeling $\sigma_{x_1} , \sigma_{x_2} , \ldots , \sigma_{x_k}$ will become $\sigma_{y_1} , \sigma_{y_2} , \ldots , \sigma_{y_k}$. The two different cases are treated as follows:

Case I : $S \cap \{ 1 , n - q \} = \emptyset$

$S - A \ := \ A - S$.

Note that if $T \in C_n^q$ is obtained as a result of Case I, then the vertices 1 and $n - q$ cannot be adjacent in T . Our construction will guarantee that in Case II below, these two vertices will always end up adjacent in the resulting q-tree.

Case II : $S \cap \{ 1 , n - q \} \neq \emptyset$

This case is split into three subcases depending on whether or not $S \cap \{ 1 , n - q \} = \{ 1 \} , \{ n - q \}$, or $\{ 1 , n - q \}$.

Subcase II A : $S \cap \{ 1 , n - q \} = \{ 1 \}$

STEP 1 : $\{ 1 \} \ := \ \{ t \}$,

STEP 2 : $S - A - \{ 1 \} \ := \ A - S - \{ t \}$,

where $t = \max (A - S)$.

Subcase II B : $S \cap \{ 1 , n - q \} = \{ n - q \}$

STEP 1 : $\{ 1 \} \ := \ \{ t \}$,

STEP 2 : $\{ n - q \} \ := \ \{ 1 \}$,

STEP 3 : $S - A - \{ n - q \} \ := \ A - S - \{ t \}$,

where $t = \max (A - S)$.

Subcase II C : $S \cap \{ 1 , n - q \} = \{ 1 , n - q \}$

STEP 1 : $\{ 1 \} \ := \ \{ \max (A - S) \}$,

STEP 2 : $\{ n - q \} \ := \ \{ t \}$,

STEP 3 : $S - A - \{1, n - q\} := A - S - \{t, \max(A - S)\}$,
where $t = \max(A - S - \max(A - S))$.

The effect of these relabelings arising from Case II above on the two special K_{q+1}'s labeled 1 and $n - q$ (omitting the relabelings of the rest of the K_{q+1}'s) before and after the labels are assigned is described symbolically in Figure 18.

Figure 18

We note that after the relabeling is carried out in Case II , the labels

$$L = \{ 1 , n - q \} \cup \{ t + 1 , t + 2 , \ldots , n \} \qquad 4.1$$

are assigned to the vertices of a K_{q+1} in T . Thus they form a clique of size $n - t + 2$ in the resulting q-tree. Furthermore $L \cup \{ t \}$ does not form a clique. Thus for Case II we have the characterization

$$t = \min \{ \ i \mid \{ 1 , n - q \} \cup \{ i + 1 , i + 2 , \ldots , n \} \text{ forms a clique in } T \ \} \quad . \qquad 4.2$$

We now show that the map Γ_n^q defined above is reversible. Given $T \in C_n^q$, we can immediately distinguish between Case I and Case II by checking whether or not the vertices 1 and $n - q$ are adjacent in T . If they are, then we have to reverse the labeling assigned in Case I. To this end, first consider the q-trail joining vertex 1 to vertex $n - q$. Let ρ_k be the last K_q on this q-trail. We now perform the following operations as in the $q = 2$ case:

(i) orient the edges of ρ_k as indicated by the vertices. Thus each edge $\{ i , j \}$ with $i < j$ in ρ_k is oriented from j to i .

(ii) "bump" the labels on ρ_k out. These form the set S .

(iii) reverse the process of "pushing" by sliding the labels of the nodes incident to the vertices of K_q 's that have already been oriented to the associated K_{q+1} 's while making these vertices sources,

(iv) perform the relabeling of K_{q+1} indicated by $A - S \ := \ S - A$.

Now we assume that the vertices 1 and $n - q$ are adjacent in T . Define t as in 4.2 and L as in 4.1 . To reconstruct the pair (S , T') with $\Gamma_n^q (S , T') = T$, we first have to identify the K_q that will function as the base. This can be accomplished as follows: Among all of the q-trails joining vertex t to a vertex in L , there exists a unique one joining t to some $v \in L$, say $(\sigma_t , \rho_1 , \ldots , \sigma_{k-1} , \rho_k , \sigma_v)$, such that all of the vertices $L - \{ v \}$ are contained in ρ_k . The uniqueness of such a q-trail (and such a vertex v) is an easy consequence of Lemma 2.1. Note that this vertex v immediately tells us which one of the three Subcases II A , II B , or II C needs to be reversed. If $v = n - q$ then we are in Subcase II A , if $v = 1$ then we are in Subcase II B , otherwise we are in Subcase II C . At this point, the steps (i) - (iii) above are performed verbatim. It is tedious but straightforward to show that the relabelings given in each one of the three Subcases can then be reversed to obtain $(S , T') = (\Gamma_n^q)^{-1} (T)$.

Composing these two bijections we have

Theorem 4.1 *The map* $\Theta_n^q = \Omega_n^q \circ \Gamma_n^q$ *is a bijection between* $S_n^q \times F_n^q$ *and* C_n^q *.Thus*

$$|C_n^q| = \binom{n}{q} [\, q\, n - q^2 + 1\,]^{n-q-2} \quad .$$

We now consider the generalizations of Corollaries 3.1 and 3.2 . The main observation is that instead of the special labels 1 and $n-q$, we could have used any two labels $\{ i , j \} \subset \{ 1 , 2 , \ldots , n \}$ in the construction of Θ_n^q . Furthermore, in the definition of the special vertex t that is involved, the numerical values of the labels are not required. All that is needed is a total order on the labels.

Since under our bijection the vertices 1 and $n-q$ are adjacent if and only if $S \cap \{ 1 , n-q \} \neq \varnothing$, we immediately obtain that the number of labeled q-trees on n nodes in which vertices labeled i and j are adjacent is given by

$$\left[\binom{n}{q} - \binom{n-2}{q} \right] [\, q\, n - q^2 + 1\,]^{n-q-2} \quad . \qquad 4.3$$

Actually our bijection tells us more. Suppose at first that we are interested in counting the number of q -trees on n vertices where for a given $t \in \{ n-q+1 , n-q+2 , \ldots , n \}$, the vertices in the set L in 4.1 form a clique of size $n-t+2$ but $L \cup \{ t \}$ does not form a clique. This count depends solely on the nature of the set S and not on $f \in F_n^q$ that is involved in the construction. More precisely, we need the count of the number of subsets $S \in S_n^q$ such that

(1) $S \cap \{ 1 , n-q \} = \{ 1 \}$ and $\max (A - S) = t$,

(2) $S \cap \{ 1 , n-q \} = \{ n-q \}$ and $\max (A - S) = t$,

(3) $S \cap \{ 1 , n-q \} = \{ 1 , n-q \}$ and $\max (A - S - \max (A - S)) = t$.

corresponding to the three Subcases involved in the construction of Γ_n^q , respectively. Here $A = \{ n-q+1 , n-q+2 , \ldots , n \}$ as before. It is not difficult to verify that the number of $S \in S_n^q$ satisfying the properties (1), (2), and (3) above are

$$\binom{t-3}{q-n+t-1} , \quad \binom{t-3}{q-n+t-1} , \quad (n-t)\binom{t-3}{q-n+t-1} \ ,$$

respectively. Thus the total number of such subsets is

$$(n-t+2)\binom{t-3}{q-n+t-1} \quad . \qquad 4.4$$

If we are interested in counting the number of subsets satisfying (1), (2), and (3) above where the equality in the conditions $\max(A-S)=t$, and $\max(A-S-\max(A-S))=t$ is replaced by $\leq$, we obtain the sum

$$\sum_{i=n-q+1}^{n}(n-t+2)\binom{t-3}{q-n+t-1} \quad . \qquad 4.5$$

Note that for a given $f \in F_n^q$, 4.5 enumerates the number of subsets $S \in S_n^q$ for which the labels

$$\{1, n-q\} \cup \{t+1, t+2, \ldots, n\}$$

form a clique of size $n-t+2$ in $\Theta_n^q(S, f)$. By our remarks above, the selection of names for the vertices is immaterial as long as they are totally ordered. Letting $m = n-t+2$, we thus have

Corollary 4.1 *The number of labeled q-trees on* $n > q+1$ *nodes in which the vertices* $v_1, v_2, \ldots, v_m \in \{1, 2, \ldots, n\}$ *form a clique is*

$$\left[\sum_{i=0}^{q+1-m}(q+1-i)\binom{n+i-q-2}{i}\right]\times(nq-q^2+1)^{n-q-2} \quad .$$

Note that Corollaries 3.1 and 3.2, as well as the formula given in 4.3 for $m=2$ are special cases of Corollary 4.1 modulo routine application of binomial identities.

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