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VISIBILITY GRAPHS OF STAIRCASE POLYGONS WITH UNIFORM STEP LENGTH

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ABSTRACT

Let Γ_n denote the collection of visibility graphs of staircase polygons (orthogonal convex fans) which consist of $n-1$ horizontal steps of arbitrary lengths. We show that for $n \geq 7$ there are graphs in Γ_n which cannot be realized as the visibility graphs of staircase polygons with uniform step length.

Keywords: Visibility graph, staircase polygon, orthogonal convex fan, uniform step length.

1. Introduction

O'Rourke [9] remarks that "some of the fundamental unsolved problems involving visibility in computational geometry will not be solved until the combinatorial structure of visibility is more fully understood". In this regard, one of the central open questions is that of characterizing visibility graphs of various classes of polygons (ElGindy [4], Ghosh [7], O'Rourke [10]).

At the time of this writing, there is no polynomial algorithm known for the solution of the recognition problem for visibility graphs, i.e. to decide whether or not a given graph is the visibility graph of some polygon. Nor is this problem known to be NP-hard. In fact, it is not even known if the recognition problem is in class NP. At this point we only have Everett's [5] result that the recognition problem is in PSPACE. Given this state of affairs, current results have involved restricting the class of polygons (Everett and Corneil [6], Abello, Eğecioğlu, and Kumar [1]), or restricting the class of graphs (ElGindy [4]), or adding some extra information to the graph (Ghosh [7], O'Rourke [10]).

In this paper, we answer a question suggested by the discussion in O'Rourke [9, p.171] about visibility graphs of staircase polygons (orthogonal convex fans) with uniform step length. Namely, we show that this class is a proper subclass of the class of visibility graphs of general staircase polygons. We obtain this result by first noticing that the requirement of uniform step length makes the visibility problem amenable to a linear programming formulation (section 3) and then show that a certain system of linear inequalities is infeasible (section 4). In the last section of the paper (section 5) we elaborate on how we came about the counterexample and we discuss also some of the implications of the linear programming approach.

2. Visibility Graphs and Staircase Polygons

Let $G = (V, E)$ be an undirected graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. Since the adjacency matrix A_G of G is symmetric with zero diagonal, G is completely characterized by the array L_G of the subdiagonal entries of A_G. To simplify our exposition here, we shall refer to L_G as the *matrix* of G (see Figure 3). We shall denote by $L_G(i, j)$ the entry in *column* i *and row* j of L_G. Note that in this setting, the columns and the rows of L_G are indexed by $i = 1, 2, \ldots, n-1$ from left to right and $j = 2, 3, \ldots, n$ from top to bottom, respectively.

The notion of visibility that we deal with is closely related to the "clear visibility" of Breen [2]. It differs from the generally used notion in that we do not allow visibility lines to go through intermediate vertices. More specifically, given a polygon P in the plane, we say that two of its vertices i and j are *visible* or *see* each other if the line segment joining i and j is a boundary segment of P, or it is completely contained in the interior of P. We use this non-standard qualification since we believe that the essential questions regarding visibility graphs remain unchanged when we adhere to this notion.

The *visibility graph* of P is then the graph

$$VG(P) = (\ \textit{Vertices of } P\ ,\ \{(i, j) \mid i \textit{ sees } j\}).$$

Consider a monotone non-increasing path π consisting of horizontal and vertical line segments that connects a point p on the positive y-axis to a point q on the positive x-axis. If π consists of a finite number of line segments, then it is called a *staircase path.* A *staircase polygon* or an *orthogonal convex fan* S is a staircase path π together with the portion from the origin to p of the y-axis, and the portion from the origin to q of the x-axis. If we assume that the horizontal segments of π have lengths $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1})$ from left to right, and the vertical segments have lengths $\mathbf{y} = (y_1, y_2, \ldots, y_{n-1})$ from top to bottom, then S is called a staircase polygon of order n. Thus a staircase polygon of order n is completely determined by a pair of positive vectors $\mathbf{x}, \mathbf{y} \in \mathcal{R}^{n-1}$ with $p = \sum_{i=1}^{n-1} y_i$, and $q = \sum_{i=1}^{n-1} x_i$. When S has order n, $VG(S)$ is a Hamiltonian graph on $2n$ vertices.

Given a staircase polygon of order n, let u_0 denote the origin, and label the $2n-1$ vertices on π from $v_1 = (0, p)$ to $v_n = (q, 0)$ in a clockwise fashion by

$$v_1,\ u_1,\ v_2,\ u_2, \ldots, v_i,\ u_i,\ v_{i+1}, \ldots, u_{n-1},\ v_n\ , \qquad (1)$$

as shown on the staircase polygon in Figure 1.

Figure 1: A staircase polygon.

Some elementary properties of the visibility graphs of staircase polygons are given in the following proposition:

Proposition 1 *Suppose $G = VG(S)$ is the visibility graph of a staircase polygon S of order $n > 3$. Then*

1. *u_0 is the only vertex of G of degree $2n-1$,*
2. *In $G \setminus u_0$, the neighborhood of u_i consists of $\{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, n-1$,*
3. *In $G \setminus u_0$, the neighborhood of v_i contains the vertices $\{v_{i-1}, u_{i-1}, u_i, v_{i+1}\}$ for $i = 2, 3, \ldots, n-1$. Furthermore u_{i-1} and u_i are the only u_j's adjacent to v_i,*
4. *In $G \setminus u_0$, the vertices $\{u_1, v_2\}$ and $\{u_{n-1}, v_n\}$ are contained in the neighborhoods of v_1 and v_n, respectively. Furthermore, u_1 is the only u_j adjacent to v_1 and u_k is the only u_j incident to v_n.*

Proof. Omitted. □

The subgraph $G(S)$ of $VG(S)$ induced by the vertices $\{v_1, v_2, \ldots, v_n\}$ will be called the *staircase subgraph* of $VG(S)$. More generally, a given labeled graph on $\{v_1, v_2, \ldots, v_n\}$ is called a *staircase graph* if $v_1 \to v_2 \to \cdots \to v_n$ is a Hamiltonian path in G, and G is isomorphic to $G(S)$ for some staircase polygon S in which the labeling of the vertices of the staircase path π has the ordering in 1. We let Γ_n denote the collection of staircase graphs on n vertices. Note that the graphs in Γ_n are labeled.

The study of $VG(S)$ is equivalent to the study of $G(S)$ in the following sense: from a given unlabeled graph $G = (V, E)$ which is isomorphic to some $VG(S)$, the staircase subgraph $G(S)$ can unambiguously be constructed modulo renaming of the axes.

Proposition 2 *Suppose $G = (V, E)$ is isomorphic to some $VG(S)$. Then the staircase subgraph $G(S)$ of $VG(S)$ can be found with $O(|V| + |E|)$ operations.*

Proof. We have that $|V| = 2n$ for some $n \geq 1$. Without loss of generality assume $n > 3$. Locate the unique vertex u_0 of G of degree $2n - 1$ and construct the graph $G \setminus u_0$. Partition the vertex set of $G \setminus u_0$ into two parts U_1 and V_1 where U_1 consists of all vertices in $G \setminus u_0$ of degree exactly two, and V_1 consists of all vertices of degree larger than two. There are three cases to consider according to if $|U_1| = n-1, n,$ or $n+1$.

If $|U_1| = n-1$, then U_1 actually consists of the vertices $u_1, u_2, \ldots, u_{n-1}$ in some order. By part 4 of Proposition 1, there exists exactly two vertices in V_1 each of which is adjacent to only one vertex in U_1. We may take v_1 to be one of these vertices. Assume v_1 is adjacent to $u_1 \in U_1$. Next, let v_2 be the common neighbor of v_1 and u_1. By part 2 of Proposition 1, v_2 is adjacent to a vertex other than u_1 in U_1. Call this vertex u_2. Now the common neighbor of v_2 and u_2 gives v_3. Continuing in this manner, the vertices $v_1, v_2, \ldots, v_n$ can be identified *in order*. Note that the ordering $v_1, v_2, \ldots, v_n$ is unique up to an isomorphism exchanging v_i and v_{n+1-i}.

Now assume $|U_1| = n$. By parts 3 and 4 of Proposition 1, U_1 consists of $u_1, u_2, \ldots, u_{n-1}$ in some order, together with one of v_1 or v_n, say

$$U_1 = \{u_1, u_2, \ldots, u_{n-1}, v_1\} \quad .$$

Furthermore, by part 4 of Proposition 1, v_1 is adjacent to exactly one vertex in U_1, which we take to be u_1. Note that in this case v_1 and u_1 are indistinguishable. The identification of v_2 through v_n can now be carried out exactly as above.

In case $|U_1| = n+1$, we can similarly identify the pairs $\{v_1, u_1\}$, and $\{v_n, u_{n-1}\}$, as these two vertices $v_1, v_n \in U_1$ are adjacent to single vertices in U_1, which we take to be u_1 and u_{n-1} respectively. We may then follow the steps of the case $|U_1| = n-1$ above to identify the vertices $v_1, v_2, \ldots, v_n$ of $G(S)$.

In any case, the above procedure returns the induced subgraph $G(S)$ on the vertices $\{v_1, v_2, \ldots, v_n\}$.

We omit the proof that the above algorithm can be implemented in linear time. □

Note that from the three cases of Proposition 2 it follows that the automorphism group of $VG(S)$ for a staircase polygon of order $n > 3$ is one of $\mathcal{Z}_2$, $\mathcal{Z}_2 \times \mathcal{Z}_2$, or $\mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{Z}_2$, where $\mathcal{Z}_2$ is the cyclic group of two elements.

3. Encoding of Staircase Graphs

We first provide a coordinate free combinatorial construction to encode staircase graphs. Given a staircase polygon S defined by $\mathbf{x}, \mathbf{y} \in \mathcal{R}^{n-1}$, $\mathbf{x}, \mathbf{y} > 0$, let $G = G(S)$. For notational convenience, denote the vertex v_i simply by i for $i = 1, 2, \ldots, n$. For $1 \leq i < j \leq n$ let s_{ij} denote the negative of the slope of the line segment through the vertices i and j. Thus

$$s_{ij} = \frac{y_i + y_{i+1} + \ldots + y_{j-1}}{x_i + x_{i+1} + \ldots + x_{j-1}}. \tag{2}$$

Suppose we arrange the slopes s_{ij} in the form of a triangular array as shown in Figure 2. Vertex i is visible from vertex j in S if and only if $s_{ik} < s_{ij}$ for every

Figure 2: Triangular array of slopes for the polygon in Figure 1.

vertex k with $i < k < j$. In other words i sees j if and only if the entry in column i and row j of this array is strictly larger than all of the entries above it in the same column.

Thus the matrix of the staircase graph G is obtained from this array of slopes after replacing s_{ij} by 1 if $s_{ik} < s_{ij}$ for every $i < k < j$, and replacing s_{ij} by 0 otherwise (see Figure 2). Note that we automatically have $L_G(i, i+1) = 1$ for $1 \leq i < n$.

Suppose now we are given a triangular array L_G corresponding to an arbitrary undirected graph G in which $L_G(i, i+1) = 1$ for $1 \leq i < n$. Let M_s denote the collection of strict inequalities

$$s_{ik} - s_{ij} < 0 \tag{3}$$

for $i < k < j$ in the components of $\mathbf{x}$ and $\mathbf{y}$, obtained from entries $L_G(i,j) = 1$ in L_G. For every $L_G(i,j) = 0$, let k be the largest index in the range $i < k < j$ for which $L_G(i,k) = 1$. Note that such an index k always exists since $L_G(i, i+1) = 1$ for every i. Geometrically k corresponds to the first vertex counterclockwise of vertex j that sees vertex i. Set

$$s_{ij} - s_{ik} \leq 0. \tag{4}$$

Denote by M_w the set of weak inequalities of the form 4, each obtained from a zero entry of L_G. G is a staircase graph if and only if the nonlinear system of inequalities $M_s \cup M_w$ has a feasible solution $\mathbf{x}, \mathbf{y} \in \mathcal{R}^{n-1}$; $\mathbf{x}, \mathbf{y} > \mathbf{0}$.

Figure 3: The matrix L_G.

Suppose now that there is a realization of a staircase graph G in which the step lengths x_i are uniform. Without loss of generality, we may assume that G has a geometric realization in which $\mathbf{x} = (1, 1, \ldots, 1)$. Then in such a realization, $x_i + x_{i+1} + \cdots + x_{j-1} = j - i$, and from 3 and 4 it follows that $y_1, y_2, \ldots, y_{n-1}$ must satisfy $|M_s| = \sum_{1 \leq i < j \leq n} (j - i - 1) L_G(i,j)$ strict inequalities of the form

$$(j-k)\,[y_i + y_{i+1} + \cdots + y_{k-1}] \;-\; (k-i)\,[y_k + y_{k+1} + \cdots + y_{j-1}] \;<\; 0, \qquad (5)$$

and $|M_w| = \sum_{1 \leq i < j \leq n} (1 - L_G(i,j))$ weak inequalities of the form

$$(k-j)\,[y_i + y_{i+1} + \cdots + y_{k-1}] \;+\; (k-i)\,[y_k + y_{k+1} + \cdots + y_{j-1}] \;\leq\; 0, \qquad (6)$$

which are obtained from 2, 3, and 4 after regrouping various terms.

Thus theoretically, we can check whether or not a given undirected, unlabeled graph H is isomorphic to $VG(S)$ for some staircase polygon S with uniform step length by going through the following steps. Without loss of generality, we assume that H has at least 8 vertices.

1. If H has $2n$ vertices for some n and the procedure in Proposition 2 returns $G = G(S)$ on the vertex set $\{v_1, v_2, \ldots, v_n\}$, then go to Step 2. Else, if any step of the procedure in Proposition 2 fails, then H is not isomorphic to $VG(S)$ for any staircase polygon S.
2. Construct the matrix L_G using the ordering $v_1 \to v_2 \to \cdots \to v_n$ produced in Step 1. Go to Step 3.
3. Check whether or not the feasibility problem which consists of $\sum_{i<j} (j - i - 1) L_G(i,j)$ strict inequalities and $\sum_{i<j} (1 - L_G(i,j))$ weak inequalities that are obtained from the entries in L_G has a solution. H is isomorphic to the visibility graph of a staircase polygon with uniform step lengths if and only if this feasibility problem has a solution.

4. A Negative Result

Theorem 1 *The visibility graph of the staircase polygon in Figure 1 cannot be realized with uniform step length.*

Proof. To prove the theorem we shall show that the feasibility problem arising from the staircase graph G given in Figure 3 does not have a solution. For this particular G, $|M_s| = 25$ and $|M_w| = 6$. The resulting $|M_s| + |M_w| = 31$ linear constraints that are obtained from 3 and 4 above are depicted in Figure 4.

Figure 4: Inequalities for the graph G of Figure 3.

In Figure 4, the numerical label in parentheses indicates the column from which the corresponding inequality arises.

Consider the six linear forms $E_1, E_2, \ldots, E_6$ that are marked in Figure 4. These satisfy the inequalities

$$\begin{aligned}
E_1 &= 2y_1 - y_2 - y_3 < 0 \\
E_2 &= y_1 + y_2 + y_3 - 3y_4 < 0 \\
E_3 &= 3y_3 - y_4 - y_5 - y_6 < 0 \\
E_4 &= -y_1 + y_2 \leq 0 \\
E_5 &= -2y_2 - 2y_3 + 2y_4 + 2y_5 \leq 0 \\
E_6 &= -y_5 + y_6 \leq 0 \, .
\end{aligned}$$

We find that

$$4\,E_1 + E_2 + 3\,E_3 + 9\,E_4 + 3\,E_5 + 3\,E_6 = 0$$

and therefore we obtain the inequality $0 < 0$. In particular the set of inequalities in Figure 4 has no feasible solution $\mathbf{y}$. Consequently, the visibility graph of the staircase polygon in Figure 1 cannot be realized by a staircase polygon with unit horizontal steps. □

5. Conclusion and Remarks

In this section we elaborate on how we came about the counterexample of Figure 3, and some of the possible consequences of the linear programming approach.

First of all, a given staircase graph G is realizable by a staircase polygon with unit horizontal steps if and only if the system of $|M_s| + |M_w|$ inequalities (such as the system in Figure 4) arising from its matrix has a feasible solution. Using standard linear programming techniques (see for example Chvátal [3], Luenberger[8], or Schrijver [11]), we add $|M_s|$ slack variables to obtain an augmented matrix $\mathbf{M}$ from this system. After that, instead of solving the feasibility problem $\mathbf{My} \leq \mathbf{0}$, $\mathbf{y} > \mathbf{0}$, we consider the optimization problem

$$\text{(P)} \; : \quad \begin{aligned} &\text{maximize } \rho \\ &\text{s.t. } \mathbf{My} \leq \mathbf{0}, \\ &\qquad \mathbf{y} \geq \rho\,\mathbf{e}\,, \\ &\qquad \mathbf{y} \geq \mathbf{0}, \end{aligned}$$

where ρ is a new variable and $\mathbf{e}$ is the $|M_s|+|M_w|$ dimensional vector of ones. With this formulation the optimal value of (P) is 0 if and only if there exists no $\mathbf{y} > \mathbf{0}$ satisfying $\mathbf{My} \leq \mathbf{0}$. Otherwise the optimal value is unbounded.

Now consider a staircase graph G on n vertices $\{1, 2, \ldots, n\}$ in which $1 \to 2 \to \cdots \to n$ is a Hamiltonian path. Suppose we have three vertices $i < j < k$ such that i sees both j and k, but does not see any vertex y with $j < y < k$. Then

1. No vertex x with $i < x < j$ sees any vertex y with $j < y < k$,
2. Vertex j sees vertex k.

This property of staircase graphs are depicted in Figure 5.

On the matrix L_G of G these correspond to the following property

Figure 5: Geometric persistence.

Given $i < j < k$, if $L_G(i,j) = L_G(i,k) = 1$ and $L_G(i,y) = 0$ for every y in the range $j < y < k$, then

1. $L_G(x,y) = 0$ for every x with $i < x < j$, and every y with $j < y < k$,
2. $L_G(j,k) = 1$.

A matrix L_G with $L_G(i, i+1) = 1$ for $i = 1, 2, \ldots n$, which satisfies these two properties for every triple $i < j < k$ is called *persistent*. Persistency is depicted graphically in Figure 6. As an example, the matrix L_G depicted in Figure 3 is persistent. In fact by our arguments above, L_G is persistent for every staircase graph G.

Let p_n and $s_n = |\Gamma_n|$ denote the number of persistent matrices of size n and the number of staircase graphs on n vertices, respectively. Since persistency of L_G is a necessary condition for staircase graphs, $s_n \leq p_n$ for all n. Using the simple condition for persistency, we can computer-generate all persistent 0-1 matrices for small values of n. We find that $p_1 = p_2 = 1$, $p_3 = 2$, $p_4 = 6$, $p_5 = 25$, $p_6 = 138$, and $p_7 = 972$. In addition to generating persistent matrices, we have written code to generate the parameters of the linear programming problem (P) corresponding to every persistent matrix of size $n \leq 7$. When n is in this range, the optimal value of (P) found by the simplex method turns out to be unbounded for every LP problem generated, except for the persistent matrix L_G of our example in Figure 3. Therefore for $n < 7$, all staircase graphs are realizable as the visibility graphs of staircase polygons with unit horizontal steps. For $n = 7$, L_G in Figure 3 turns out to be the *only* persistent matrix which is not the visibility graph of a staircase polygon with unit horizontal steps (i.e. 1 out of 972 LP problems solved). However,

Figure 6: Persistence.

this graph is the staircase graph corresponding to the polygon in Figure 1. Thus $s_n = p_n$ for $n \leq 7$.

It can be proved that the persistency condition completely characterizes the class of staircase graphs. This and related results on recognition of visibility graphs of staircase polygons, and the combinatorics of staircase graphs can be found in Abello, Eğecioğlu, and Kumar [1].

Now Γ_n is naturally embedded in Γ_{n+1} by mapping

$$\begin{aligned} (x_1, x_2, \ldots, x_{n-1}) &\rightarrow (x_1, x_2, \ldots, x_{n-1}, 1) \\ (y_1, y_2, \ldots, y_{n-1}) &\rightarrow (y_1, y_2, \ldots, y_{n-1}, \epsilon) \end{aligned}$$

where ϵ is picked so that $\epsilon < s_{ij}$ for every slope s_{ij} given by 2. Under this embedding, a staircase graph $G = (V, E) \in \Gamma_n$ with $V = \{v_1, v_2, \ldots, v_n\}$ is embedded in Γ_{n+1} as $G' = (V', E')$ where $V' = V \cup \{v_{n+1}\}$ and $E' = E \cup \{(v_n, v_{n+1})\}$. Thus the visibility graphs of staircase polygons for $n \geq 7$ cannot in general be realized by staircase polygons with unit horizontal steps, since a copy of the smallest counterexample G lives in each Γ_n for $n \geq 7$.

However, our negative result does not provide a geometric intuition as to *why* uniform steps do not work for our particular example G of Figure 3. It would be desirable to characterize those staircase graphs realizable by uniform step lengths by a formulation that does not require linear programming, and possibly explaining whether or not G is in effect the only obstruction for non constructibility by uniform step lengths.

Linear programming formulation may prove useful for visibility properties of larger classes of polygons, and also in optimization of other interesting properties of staircase graphs in particular, and larger families of convex fans in general. For example, it is tempting to conjecture that *two* distinct horizontal step lengths should allow for the representation of all visibility graphs of staircase polygons.

We note that a polynomial time solution to the recognition and reconstruction problem for visibility graphs of staircase polygons have recently been found (Abello, Eğecioğlu, and Kumar [1]). These problems for convex fans were previously explored by ElGindy [4] with inconclusive results.

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