(Example from pg 197) Prove the following equation holds for all $n \in \mathbb{Z}^+$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Basis Step : $n = 1$

$$\sum_{i=1}^{n} i^2 = 1^2 = 1$$

$$\frac{1 \cdot (1+1) \cdot (1 \cdot 2 + 1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6} = 1$$

2. Inductive Step
   Assuming that the following equation is true (for n)

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

   Show that this equation holds (for $n+1$)

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \tag{2}$$

   First simplify Equation 2.

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}\end{aligned}$$

   Now for the proof:

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 = \left(\sum_{i=1}^{n} i^2\right) + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\ &= \frac{(n+1)(2n^2 + n + 6n + 6)}{6} \\ &= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(2n+3)(n+2)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}\end{aligned}$$

(pg 219, Question 11) Define the integer sequence $a_0, a_1, a_2, ...$ recursively by

1. $a_0 = 1$, $a_1 = 1$, $a_2 = 1$; and
2. $a_n = a_{n-1} + a_{n-3}$ for $n \geq 3$

Prove the following holds for all $n \in \mathbb{N}$

$$a_{n+2} \geq (\sqrt{2})^n$$

Proof by Mathematical Induction (Alternate Form pg 206):

1. Basis Step
   We need to show the basis step for $n = 0$, $n = 1$, and $n = 2$.
   $n = 0 : a_{0+2} = a_2 = 1 \geq (\sqrt{2})^0 = 1$
   $n = 1 : a_{1+2} = a_3 = a_2 + a_0 = 1 + 1 = 2 \geq (\sqrt{2})^1$
   $n = 2 : a_{2+2} = a_4 = a_3 + a_1 = 2 + 1 = 3 \geq (\sqrt{2})^2 = 2$
   Why only use these for basis step? According to the Alternate Form of the Principle of Mathematic Induction, for an open statement $S(n)$ and 2 positive integers n_0 and n_1, $n_0 \leq n_1$

   - If $S(n_0), S(n_0 + 1), S(n_0 + 2), ..., S(n_1 - 1), S(n)$ are all true; and
   - If whenever $S(n_0), S(n_0 + 1), ...S(k - 1), S(k)$ are true for some $k \in \mathbb{Z}^+$, then $S(k + 1)$ is also true.

   then $S(n)$ is true for all $n \geq n_0$.

   So, in our case, $n_0 = 0$ and $n_1 = 2$. Since we have proven the inequality holds for $n = 0$ and $n = 2$ by brute force, we are able to prove it for $n = 3$, and then use that to prove it for $n = 4$, and so on.

2. Induction Assuming for some $k \in \mathbb{N}, k > 2$, $a_{k+2} \geq (\sqrt{2})^k$, show that the following inequality holds.

$$a_{(k+1)+2} \geq (\sqrt{2})^{k+1} \tag{3}$$

   First, I simplify Equation 3a little, to make things easier later on.

$$\begin{aligned} a_{(k+1)+2} &\geq (\sqrt{2})^{k+1} \\ a_{k+3} &\geq (\sqrt{2})^{k+1} \\ a_{k+3} &\geq (\sqrt{2})(\sqrt{2})^k \end{aligned}$$

   Now, I use the recursive definition to prove the inequality

$$\begin{aligned} a_{k+3} = a_{k+2} + a_k &\geq (\sqrt{2})^k + (\sqrt{2})^{k-2} \\ &\geq (\sqrt{2})^{k-2}((\sqrt{2})^2 + 1) \\ &\geq (\sqrt{2})^{k-2}(2 + 1) \\ &\geq (\sqrt{2})^{k-2} \cdot 3 \\ &\geq (\sqrt{2})^k \cdot (\sqrt{2})^{-2} \cdot 3 \\ &\geq (3/2) \cdot (\sqrt{2})^k \end{aligned}$$

So, we see that the Equation 3 holds.

$$a_{k+3} \geq (3/2)(\sqrt{2})^k \geq (\sqrt{2})(\sqrt{2})^k$$
$$a_{k+3} \geq (\sqrt{2})^{k+1}$$