Linear Programming

- **LP** is a general method to solve **optimization** problem with **linear** objective function and linear constraints.
- **Diet Problem Example:** Feeding an army.
- 4 menu choices: **Fish, Pizza, Hamburger, Burrito.**
- Each choice has **benefits** (nutrients) and a **cost** (calories).

	A	C	D	Calories
F	200	90	100	600
P	75	80	250	800
H	275	80	510	550
B	500	95	200	450
RDA	2000	300	475	

- **Determine least-caloric diet that satisfies RDA of nutrients.**

LP Formulation

- **Introduce variables x_F, x_P, x_H, x_B for the amount (units) consumed of each food.**
- **Obj. Function is total calories.**

  minimize $600x_F + 800x_P + 550x_H + 450x_B$

- **Each of the nutrient constraint can be expressed as a linear function:**

$$\begin{aligned} 200x_F + 75x_P + 275x_H + 500x_B &\geq 2000 \\ 90x_F + 80x_P + 80x_H + 95x_B &\geq 300 \\ 100x_F + 250x_P + 510x_H + 200x_B &\geq 475 \end{aligned}$$

- **Finally, all food quantities consumed should be non-negative:**

$$x_F \geq 0; \; x_P \geq 0; \; x_H \geq 0; \; x_B \geq 0$$

- **This is an example of Linear Prog.**

Manufacturing Example

- **A plant has 3 types of machines, A, B, C.**
- **Choice of 4 products: 1, 2, 3, 4.**
- **Resource consumption (how much machine time a product uses) and profit per unit.**

	A	B	C	Profit
1	2	0.5	1.5	3.5
2	2	2	1	4.2
3	0.5	1	3	6.5
4	1.5	2	1.5	3.8
Avail.	20/wk	30/wk	15/wk	

- **Machine shop operates 60 hrs/week.**
- **Determine most profitable product mix.**

LP Formulation

- **Introduce variables x_1, x_2, x_3, x_4 for the units of different products manufactured.**
- **Obj. Function is total profit.**

$$\textbf{maximize} \quad 3.5x_1 + 4.2x_2 + 6.5x_3 + 3.8x_4$$

- **Each of the machines provides a constraint, expressed as a linear function:**

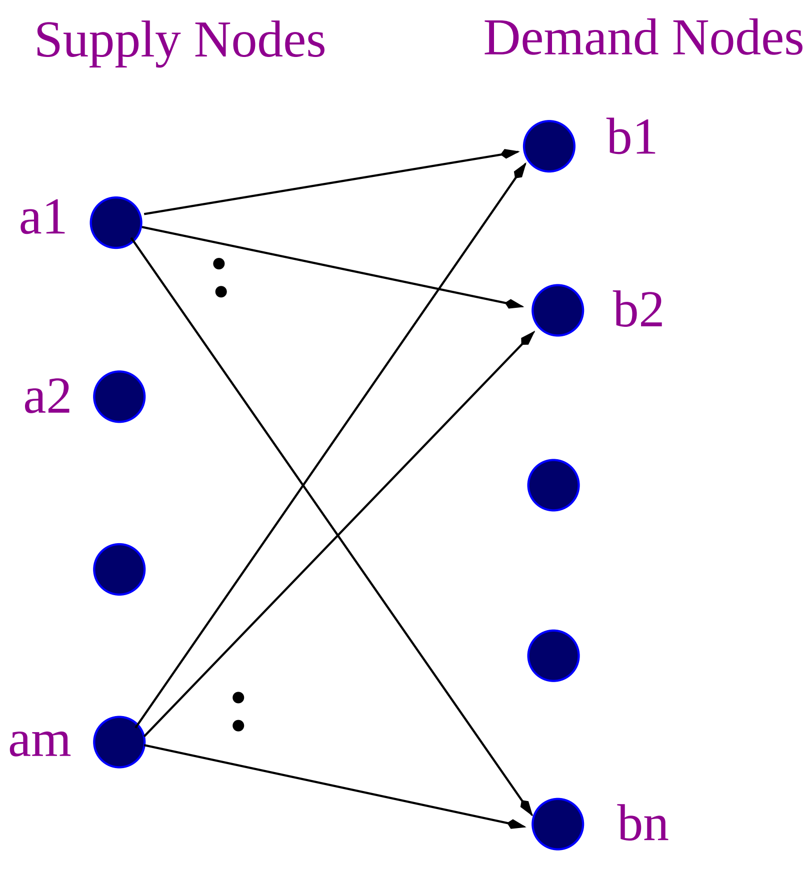
$$\begin{aligned} 2x_1 + 2x_2 + 0.5x_3 + 1.5x_4 &\leq 1200 \\ 0.5x_1 + 2x_2 + x_3 + 2x_4 &\leq 1800 \\ 1.5x_1 + x_2 + 3x_3 + 1.5x_4 &\leq 900 \end{aligned}$$

- **Finally, all product quantities should be non-negative:**

$$x_1, \ x_2, \ x_3, \ x_4 \geq 0$$

Transportation Example

- m **supply sites, and** n **demand sites.**
- **Supply site** i **produces** a_i **units;**
  demand site j **requires** b_j **units.**
- c_{ij} **cost of shipping one unit from** i **to** j**.**



- **Determine optimal shipping cost.**

LP Formulation

- **Let x_{ij} be the amount shipped from i to j. Then, the LP is**

$$\textbf{min} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

- **subject to**

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \quad i = 1, 2, \ldots, m \\ \sum_{i=1}^{m} x_{ij} &= b_j, \quad j = 1, 2, \ldots, n \\ \sum_{i=1}^{m} a_i &= \sum_{j=1}^{n} b_j \\ x_{ij} &\geq 0 \end{aligned}$$

General Form of LP

- n **unknowns** $x_1, \ldots, x_n$, **and** m **constraints.**

$$\textbf{max} \quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

- **subject to**

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \\ \\ x_1,\ x_2, \cdots,\ x_n &\geq 0 \end{aligned}$$

- **A company making** n **products, each requiring some amount each of** m **resources. Item** i **has profit** c_i**.**
- b_j **is total resource of type** j**.** a_{ij} **is the amount of resource** j **needed by item** i**.**

Matrix Form

- **maximize** $\quad \mathbf{cx}$
- **subject to** $\quad \mathbf{Ax} \leq \mathbf{b}$
- $\mathbf{x} \geq 0$
- **The input data are:**
  - **the** $m \times n$ **coefficient matrix A,**
  - **the** m**-vector of right hand sides b, and**
  - **the** n**-vector of objective costs c.**

Slack Variables

- **An inequality $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ can be converted to an equation by introducing a slack variable $s_i \geq 0$:**

$$a_{i1}x_1 + \cdots + a_{in}x_n + s_i \;=\; b_i$$

- **Similarly, an inequality like $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ can be written as**

$$a_{i1}x_1 + \cdots + a_{in}x_n - s_i \;=\; b_i$$

- **The Standard form Linear Program is**

  maximize $\quad \mathbf{cx}$

  subject to $\quad \mathbf{Ax} = \mathbf{b}$

  $\mathbf{x} \geq 0$

- **Slack variable coefficients in the obj function are typically zero.**

Geometrical Method

- 2-variable LP can be solved graphically.
- Useful teaching tool: Helps build intuition about the problem.
- Not a practical method for practical linear programming problems.
- We plot constraints on a rectangular coordinate system.
- A constraint is drawn as a line. The inequality constraint restricts the feasible region to one side of the line.
- The non-negative constraints $x_1, x_2 \geq 0$ restrict the feasible region to the first quadrant.

Example

- **We will consider the following LP:**

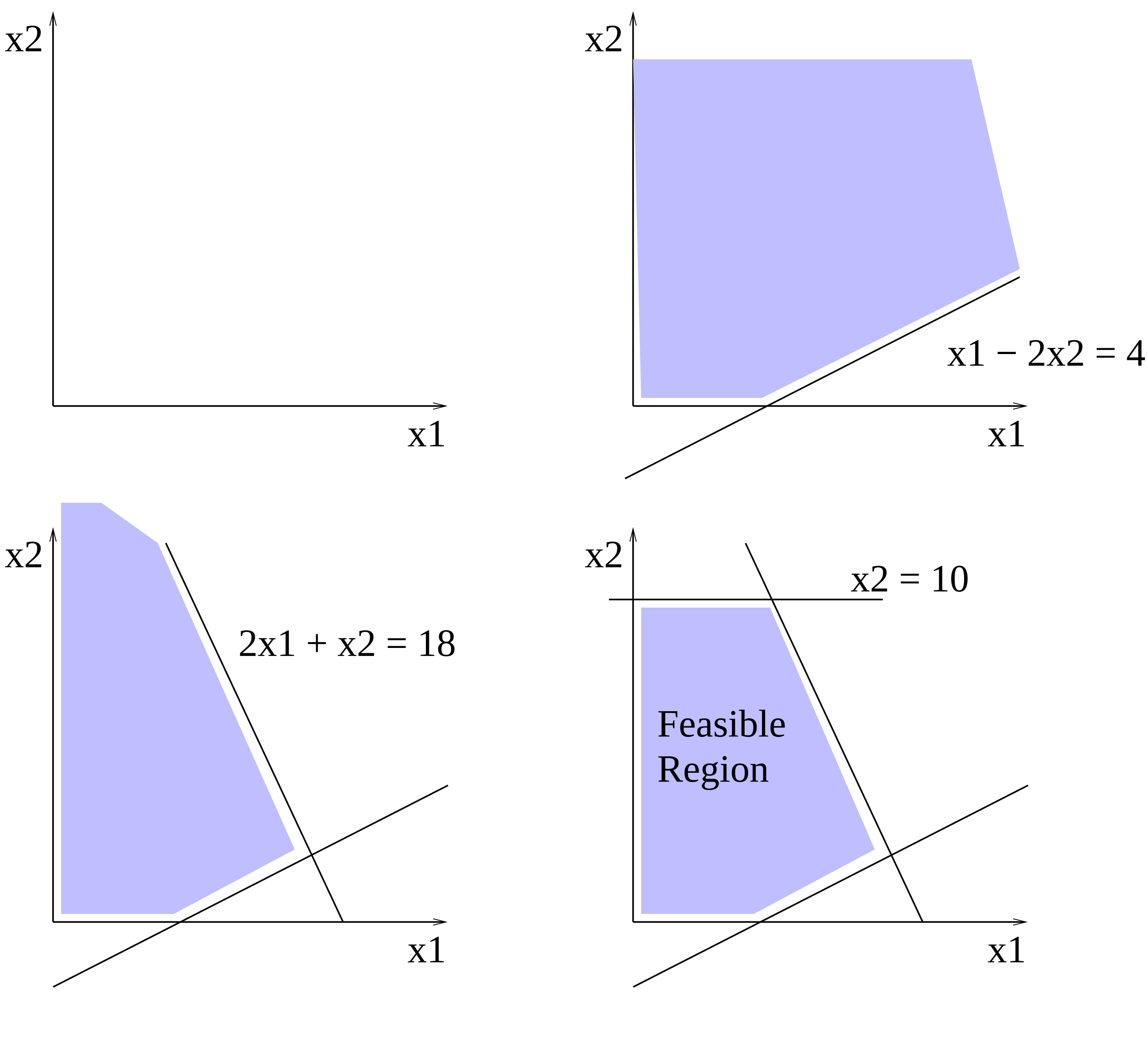
$$\textbf{max} \quad z = 2x_1 + 3x_2$$

- **subject to**

$$\begin{aligned} x_1 - 2x_2 &\leq 4 \\ 2x_1 + x_2 &\leq 18 \\ x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

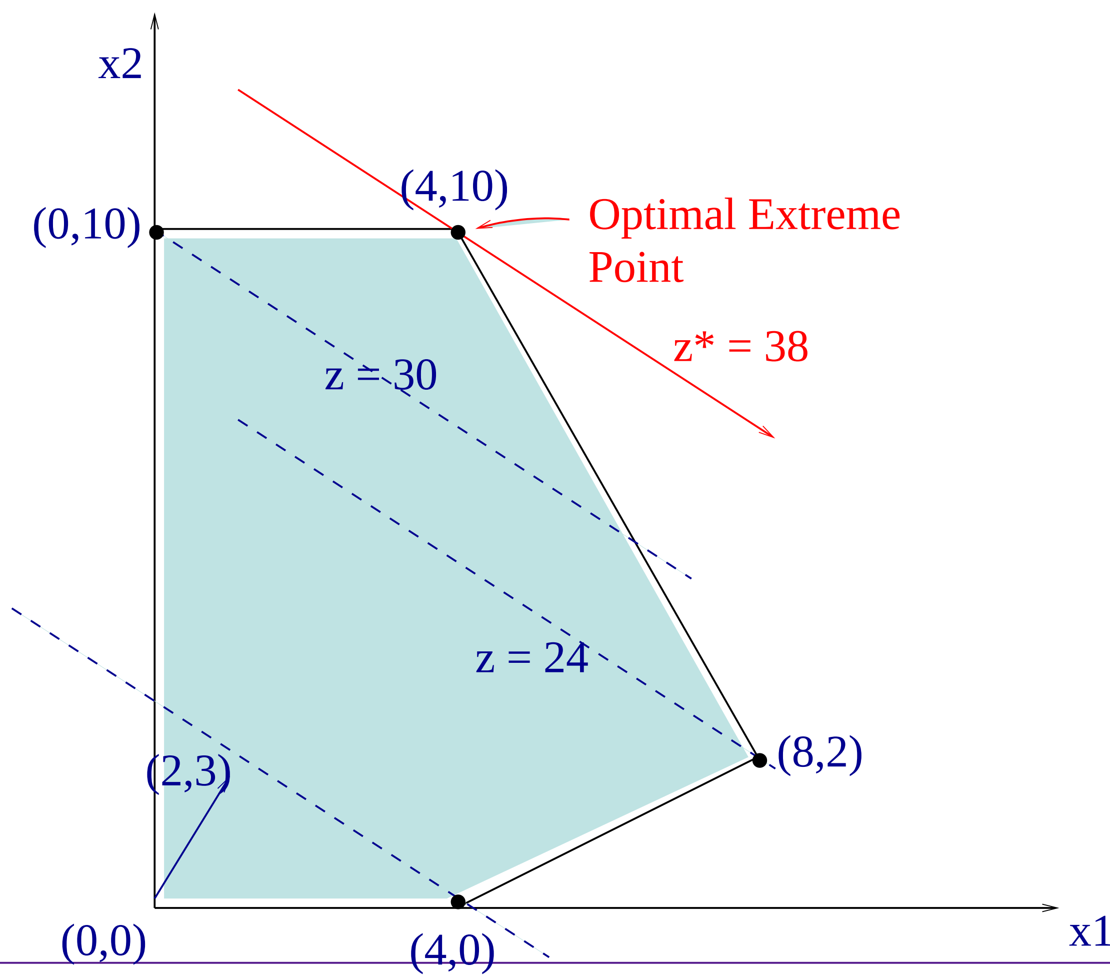
Illustration

- **Building the feasible region, one constraint at a time.**

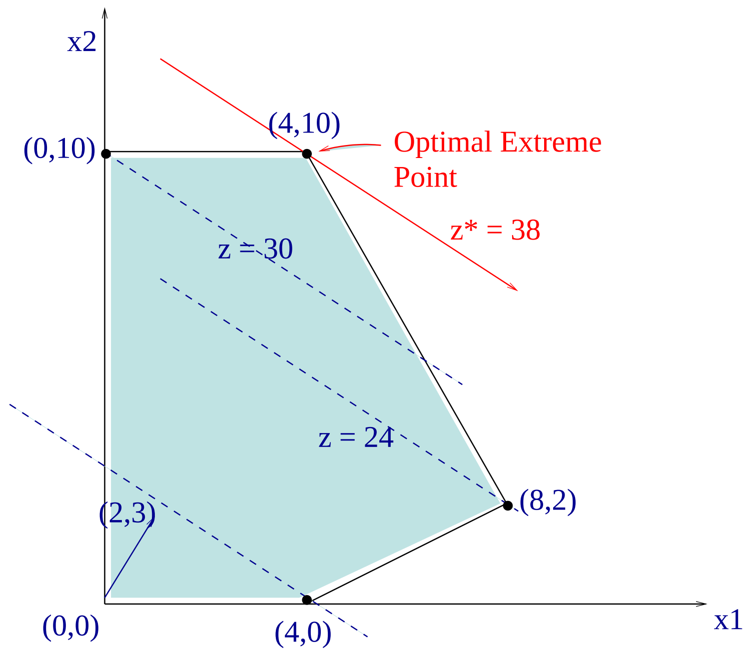


Optimal Solution

- **Examine Level curves:** $z = 1, 2$ **etc.**
- **E.g.** $z = 24$ **defines the line** $2x_1 + 3x_2 = 24$**. Similarly, for** $z = 30$**.**
- **Parallel lines–they have same slope.**
- **Line with highest** z **value that intersects feasible region.**

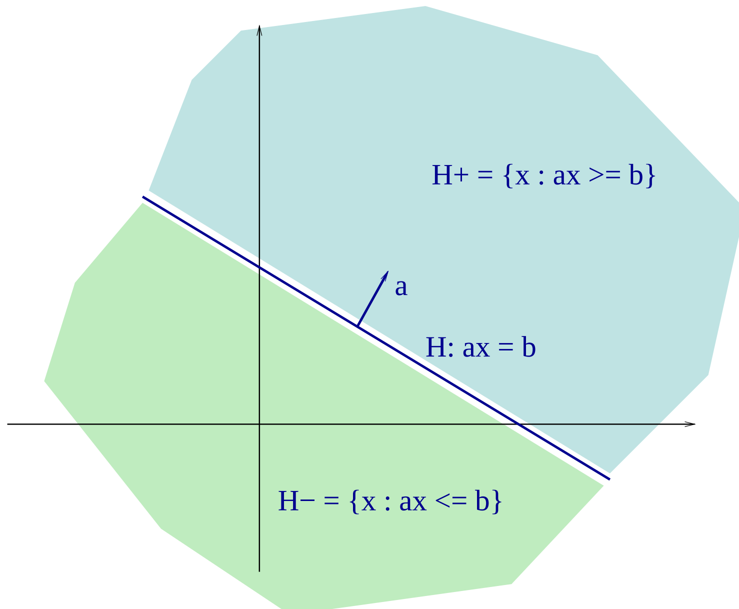


Algebra of LP



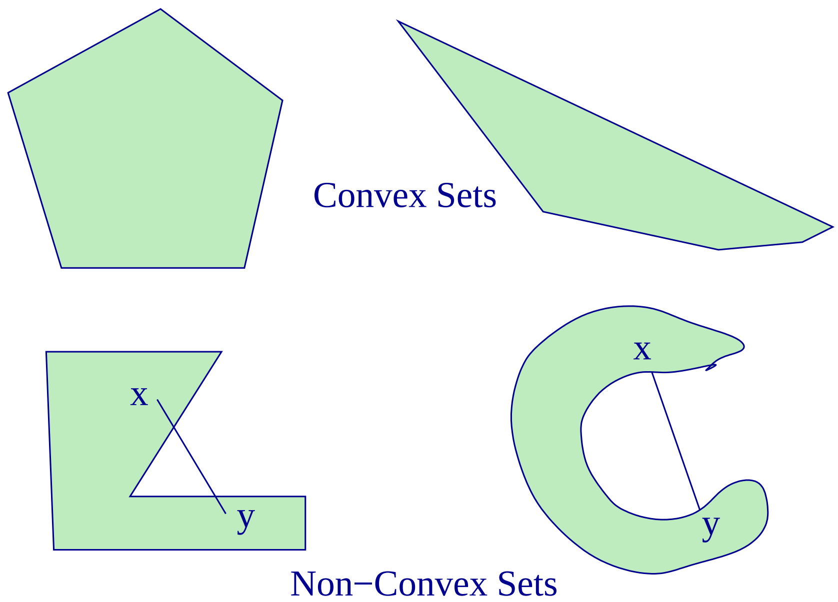
- **Important points:**
  Feasible region is convex polyhedron, optimal solution is a corner.
- **Simplex Method is a famous method for solving LP, invented by George Dantzig.**
- **Introduce algebra and geometry to explain how it works.**

Linear Algebra



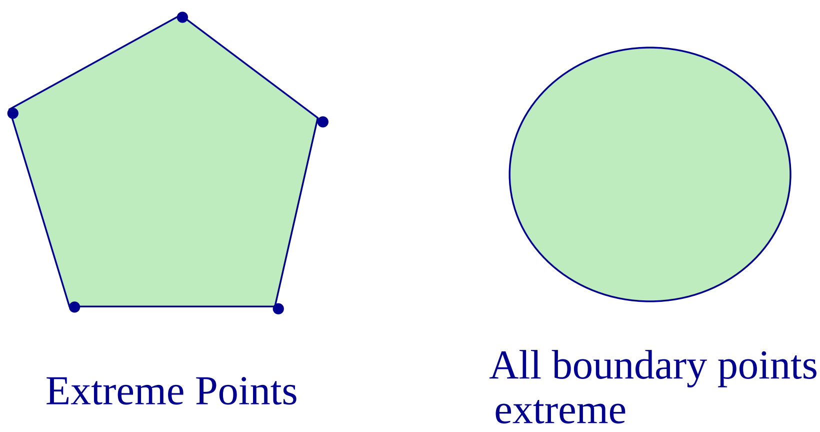
- **A hyperplane in $\mathcal{R}^n$ is the set of points $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ that satisfy $\mathbf{ax} = b$, where $\mathbf{a} \in \mathcal{R}^n$ and $b \in R$.**
- **Examples: Line $a_1x_1 + a_2x_2 = b$ in R^2.**
- **$H^+ = \{\mathbf{x} : \mathbf{ax} \geq b\}$ is the positive halfspace.**
- **Vector a is gradient of the linear function $f(\mathbf{x}) = \mathbf{ax}$, and so is normal to H.**
- **A polyhedral set is the intersection of halfspaces.**

Convexity



- **A set S is convex if for any two points $\mathbf{x}, \mathbf{y} \in S$, the line segment $\mathbf{xy}$ also lies in S.**
- **That is, $\alpha \mathbf{x} + (1 - \alpha)\mathbf{y} \in S$ if $\mathbf{x}, \mathbf{y} \in S$, for all $\alpha \in [0, 1]$.**
- **Examples: A hyperplane is convex.**
  A halfspace is convex.
  Intersection of convex sets is convex.
- **$\alpha \mathbf{x} + (1 - \alpha)\mathbf{y}$, where $\alpha \in [0, 1]$ is called convex combination of $\mathbf{x}, \mathbf{y}$.**
- **$\alpha \mathbf{x} + (1 - \alpha)\mathbf{y}$, for $\alpha \in (-\infty, \infty)$ is called linear combination of $\mathbf{x}, \mathbf{y}$.**

Extreme Points



- **A point x is extreme point of S if it is not a strict convex combination of two other points of S.**
- **Corners of a polyhedron are extreme. All boundary points of a sphere are extreme.**
- **Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, where A is $m \times n$ matrix of rank $m \leq n$.**
- **A point x is an extreme point of S iff x is the intersection of n linearly independent hyperplanes.**

Basic Feasible Solution

- **LP $\mathbf{A}\mathbf{x} = \mathbf{b}$. Assume $rank(\mathbf{A}) = m \leq n$.**
- **Partition cols as $\mathbf{A} = (B : N)$, where**
  B is $m \times m$ basis matrix
  N is $m \times (n - m)$ non-basis matrix.
- **The LP system becomes**

$$B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b}$$

- **Because B is non-singular, it is invertible:**

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_N$$

- **Set $\mathbf{x}_N = 0$. We get $\mathbf{x}_B = B^{-1}\mathbf{b}$.**
- **$\mathbf{x}_B$ is vector of basic variables,**
  $\mathbf{x}_N$ vector of non-basic vectors.
- **If $\mathbf{x}_B \geq 0$, then the solution $\mathbf{x} = \binom{B^{-1}\mathbf{b}}{0}$ is called a basic feasible solution.**

Example of BFS

- **Consider linear system:**

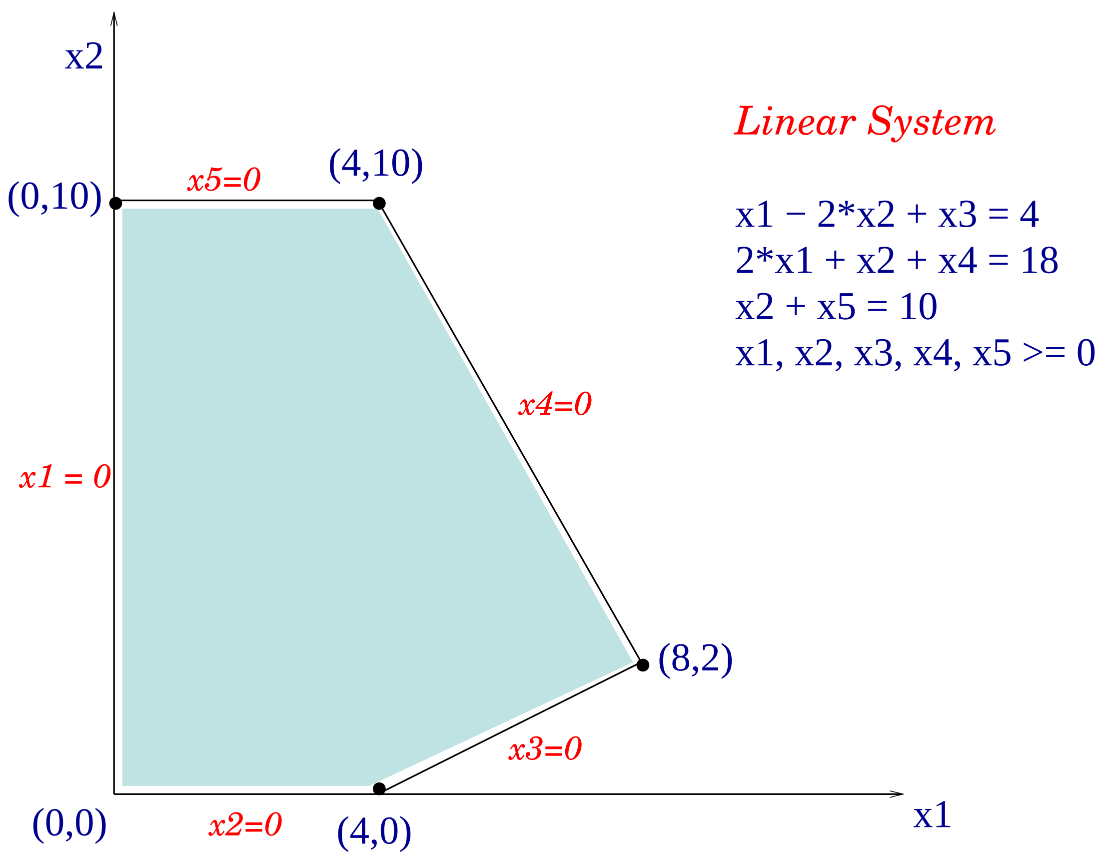
$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 15 \\ x_1 + 3x_2 + x_3 - x_5 &= 12 \end{aligned}$$

- $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 & -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$.
- **Consider basis matrix of 1st and 3rd cols:** $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.
- **Inverse matrix is** $B^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.
- **BFS is** $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix}$, **where** $\mathbf{x}_N = \mathbf{0}$, **and**

$$\begin{aligned} \mathbf{x}_B &= \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = B^{-1}\mathbf{b} \\ &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 15 \\ 12 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \end{aligned}$$

Fundamental Theorem

Theorem: **x is an extreme point of** $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ **iff x is a BFS.**



- **Each hyperplane $\leftrightarrow$ variable, which is 0 on the hyperplane.**
- **Each corner has 2 variables zero, and 3 non-zero.**

Simplex Example

- **Consider LP max** $z = 2x_1 + 3x_2$
- **subject to**

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 4 \\ 2x_1 + x_2 + x_4 &= 18 \\ x_2 + x_5 &= 10 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

- **Initial basic feasible solution:** $(x_3, x_4, x_5) = (4, 18, 10)$**, and** $(x_1, x_2) = (0, 0)$**.**
- **Objective function value is zero.**
- **Entering variable. Pick a** x_i **with** $c_i > 0$ **(max) in the objective, and make it basic. (E.g.** x_2**)**
- **Departing variable. Increase** x_1 **from zero, until some basic variable becomes zero (and non-basic).**
- **Repeat until all vars in objective function have negative coefficients.**

Simplex Example

- **Consider LP max** $z = 2x_1 + 3x_2$
- **subject to**

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 4 \\
2x_1 + x_2 + x_4 &= 18 \\
x_2 + x_5 &= 10 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

- **Initial BFS form:**

$$
\begin{aligned}
x_3 &= 4 - x_1 + 2x_2 \\
x_4 &= 18 - 2x_1 - x_2 \\
x_5 &= 10 - x_2
\end{aligned}
$$

- **Set** $x_1, x_2 = 0$**,** $x_3 = 4, x_4 = 18, x_5 = 10$**.** $z = 0$**.**
- **Entering variable is** x_2**.**
- **Max value of** x_2 **is** $\min\{\infty, 18, 10\}$
- **Departing variable is** x_5**.**

Simplex Example

- **Re-arrange the linear system:**

$$\begin{aligned} x_2 &= 10 - x_5 \\ x_3 &= 24 - x_1 - 2x_5 \\ x_4 &= 8 - 2x_1 + x_5 \end{aligned}$$

- **Set** $x_1 = x_5 = 0$**, and** $x_2 = 10, x_3 = 24, x_4 = 8$**.**
- **Obj. function becomes** $z = 30 + 2x_1 - 3x_5 = 30$**.**
- **Entering variable is** x_1**.**
- **Max value of** x_1 **is** $\min\{\infty, 24, 4\}$**.**
- **Departing variable is** x_4**.**

Simplex Example

- **Re-arrange the linear system:**

$$\begin{aligned} x_1 &= 4 - \frac{1}{2}x_4 + \frac{1}{2}x_5 \\ x_2 &= 10 - x_5 \\ x_3 &= 20 + \frac{1}{2}x_4 - \frac{5}{2}x_2 \end{aligned}$$

- **Set** $x_4 = x_5 = 0$**, and** $x_1 = 4, x_2 = 10, x_3 = 20$**.**
- **Obj. function becomes** $z = 38 - x_4 - 2x_5 = 38$**.**
- **This is optimal because all coeffs are negative.**

Complexity of LP

- **Simplex invented by G. Dantzig in 1947.**
- **One of the most popular and widely used optimization method. Many commercial software packages.**
- **Extremely fast in practice, solves LP with 10^5 constraints and variables in secs or mins.**
- **Worst-case complexity exponential!. Klee-Minty (1972)**
- **Poly-time LP algorithms in 1980's. Khachian (1979). Much worse in practice!**
- **Interior point method of Karmakar (1984) first method to be theoretically in P and comparable in practice.**
- **Open Problem: Is there a pivoting rule for which simplex is polynomial?**

LP Duality

- **Duality turns out to be a deep property of linear programming.**
- **Many famous theorems in other areas are an easy consequence of LP Duality.**
- **In order to motivate its power, we consider an investment game.**
- **Investment Game played during Mon-Fri period.**
- **Rules: If invest x one morning, and $2x$ next morning, then receive $4x$ on third morning.**
- **Amount received on day D can be invested same day for next round.**
- **Inability to pony up $2x$ on second day forfeits original x.**
- **Determine optimal investment strategy.**

Strategy 1

- **Starting fortune is \$1.**
- **Invest \$1/3 on Mon, and \$2/3 on Tues.**
- **You get \$4/3 on Wed, of which invest \$4/9.**
- **Invest remaining \$8/9 on Thurs.**
- **Collect \$16/9 on Friday.**
- **Total gain is \$7/9.**
- **Can we do better?**

Strategy 2

Mon	Tues	Wed	Thurs	Fri
1/4	1/2	(1)		
	1/4	1/2	(1)	
		1/2	1	(2)

- Total gain is $1.
- Can we do still better?
- This is a maximization LP, and we want an upper bound on the solution.
- LP duality aims to answer this question.

Illustrating Example

- **Consider LP max** $z = 4x_1 + x_2 + 5x_3 + 3x_4$
- **subject to**

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &\leq 1 \\ 5x_1 + x_2 + 3x_3 + 8x_4 &\leq 55 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 &\leq 3 \\ x_i &\geq 0 \end{aligned}$$

- **Suppose instead of computing optimal z^*, we just want a quick estimate of the max possible (upper bound).**
- **We can get lower bounds by trying feasible solutions:**

$$\begin{aligned} (0,0,1,0) &\rightarrow z^* = 5 \\ (2,1,1,1/3) &\rightarrow z^* = 15 \\ (3,0,2,0) &\rightarrow z^* = 22 \end{aligned}$$

- **However, this method still gives no estimate of an upper bound.**

Illustrating Example

- **LP max** $z = 4x_1 + x_2 + 5x_3 + 3x_4$
- **subject to**

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &\leq 1 \\ 5x_1 + x_2 + 3x_3 + 8x_4 &\leq 55 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 &\leq 3 \\ x_i &\geq 0 \end{aligned}$$

- **Multiply 2nd constraint by** $5/3$**:**

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + \frac{15}{3}x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

- **This implies** $z^* \leq \frac{275}{3}$**, because term by term this constraint dominates obj. function** z**.**
- **One more. Add constraints (2) and (3),**

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

- **This implies** $z^* \leq 58$**.**

General Strategy

- **Form linear combinations of constraints.**
- **Multiply constraint (i) by y_i, then add**
  $y_1(x_1 - x_2 - x_3 + 3x_4)+$
  $y_2(5x_1 + x_2 + 3x_3 + 8x_4)+$
  $y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \ \leq \ y_1 + 55y_2 + 3y_3$.
- **Rewrite**
  $(y_1 + 5y_2 - y_3)x_1 + \ (-y_1 + y_2 + 2y_3)x_2 +$
  $(-y_1 + 3y_2 + 3y_3)x_3 + \ (3y_1 + 8y_2 - 5y_3)x_4 \ \leq$
  $y_1 + 55y_2 + 3y_3$.
- **For this to dominate z term by term**

$$\begin{aligned} y_1 + 5y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2y_3 &\geq 1 \\ -y_1 + 3y_2 + 3y_3 &\geq 5 \\ 3y_1 + 8y_2 - 5y_3 &\geq 3 \\ y_i &\geq 0 \end{aligned}$$

- **We then have $z \leq y_1 + 55y_2 + 3y_3$.**

Dual LP

- **Our goal is to find smallest z subject to constraints on y_i.**
- **We have a linear program:**
- **minimize** $u = y_1 + 55y_2 + 3y_3$
- **subject to**

$$\begin{aligned} y_1 + 5y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2y_3 &\geq 1 \\ -y_1 + 3y_2 + 3y_3 &\geq 5 \\ 3y_1 + 8y_2 - 5y_3 &\geq 3 \\ y_i &\geq 0 \end{aligned}$$

- **This LP is called the Dual of the original LP.**

Duality Theorems

Weak Duality: **If z and u are feasible solutions of the primal and dual LPs, then**

$$z \leq u$$

- **Follows by definition of dual LP. Each feasible solution of dual is an upper bound on primal.**

Strong Duality: **If primal LP has finite optimal solution z^*, then the dual also has a finite optimal solution u^* and**

$$z^* = u^*$$

Back to Investment Game

- **Variables x_i to denote investment amounts.**

Mon	Tues	Wed	Thur	Fri
x_1	$2x_1$	$(4x_1)$		
	x_2	$2x_2$	$(4x_2)$	
		x_3	$2x_3$	$(4x_3)$

- **The initial amount is \$1.**
- **The final amount is $1 + x_1 + x_2 + x_3$.**
- **Formulate as LP: max** $1 + x_1 + x_2 + x_3$

$$
\begin{aligned}
1 - x_1 &\geq 0 \\
1 - 3x_1 - x_2 &\geq 0 \\
1 + x_1 - 3x_2 - x_3 &\geq 0 \\
1 + x_1 + x_2 - 3x_3 &\geq 0 \\
x_i &\geq 0
\end{aligned}
$$

Back to Investment Game

- **Investment game LP: max** $1 + x_1 + x_2 + x_3$
- **subject to**

$$\begin{aligned} 1 - x_1 &\geq 0 \\ 1 - 3x_1 - x_2 &\geq 0 \\ 1 + x_1 - 3x_2 - x_3 &\geq 0 \\ 1 + x_1 + x_2 - 3x_3 &\geq 0 \\ x_i &\geq 0 \end{aligned}$$

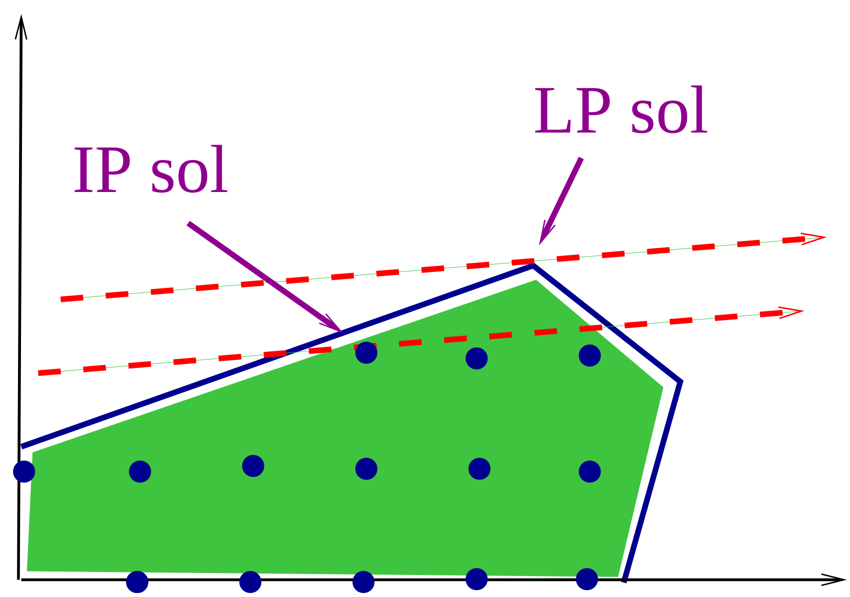
- **Linear combination** $2 * (2) + (3) + (4)$**:** $4x_1 + 4x_2 + 4x_3 \leq 4$**, which implies**

$$1 + (x_1 + x_2 + x_3) \leq 2$$

- **Thus, strategy 2 is optimal.**

Integer Programming

- maximize   **cx**
- subject to   **Ax ≤ b**
- **x** integer   (only diff from LP)
- IP feasible space is discrete, and smaller than LP feasible space, which is continuous.



Integer Programming

- **One might think that restricting variables makes problem easier to solve.**
- **In fact, IP is NP-Complete.**
- **Reduction from Boolean 3-SAT.**
- $\Phi = (x_1 \vee \overline{x_3} \vee x_4) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_4}) \wedge \cdots$
- **Boolean Satisfiability of Φ can be encoded as an IP:**
- **minimize** x_1
- **subject to**

$$
\begin{aligned}
x_1 + (1 - x_3) + x_4 &\geq 1 \\
(1 - x_2) + x_3 + (1 - x_4) &\geq 1 \\
&\vdots \\
x_i &\in \{0, 1\}
\end{aligned}
$$

Formulating IP Problems

- **Knapsack Problem:** n **items, weights** a_i**, capacity** b**.**
- **maximize** $\sum_{i=1}^{n} c_i x_i$
- **subject to**

$$\begin{array}{rcl} \sum_{i=1}^{n} a_j x_j & \leq & b \\ x_i & & \textbf{binary} \end{array}$$

- **Vertex Cover:** $G = (V, E)$
- **minimize** $\sum_{i=1}^{n} x_i$
- **subject to**

$$\begin{array}{rcl} x_i + x_j & \geq & 1, \ \textbf{for each edge } (i,j) \in E \\ x_i & & \textbf{binary} \end{array}$$

Network Problems using IP

- **Min Cost Network Flows.**
- **Given is a network $G = (V, E)$, supply b_i for each node i, capacity c_{ij} for each edge, and lower and upper bounds on flow for each edge.**
- **minimize** $\sum c_{ij} x_{ij}$
- **subject to**

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = b_i$$

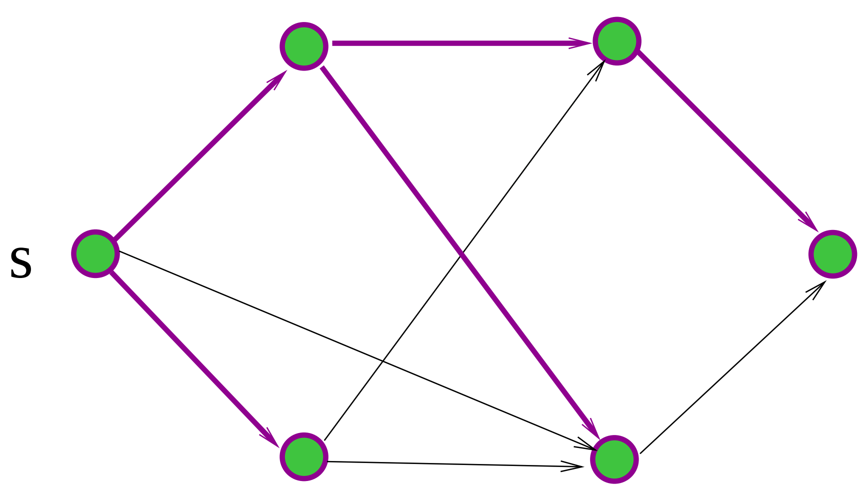
$$l_{ij} \leq x_{ij} \leq u_{ij}, \text{ for all } (i,j) \in E$$

- **No need to restrict x_{ij} to integers.**

Network Problems using IP

- **Shortest Path: 1 unit flow from s to t.**
- **minimize** $\sum c_{ij} x_{ij}$
- **subject to**

$$
\begin{aligned}
\sum_{j:(s,j)\in E} x_{sj} &= 1 \\
\sum_{j:(j,t)\in E} x_{jt} &= -1 \\
\sum x_{ij} - \sum x_{ji} &= 0 \quad \textbf{if } i \neq s, t \\
x_{ij} &\in \{0,1\} \textbf{ for all } (i,j) \in E
\end{aligned}
$$



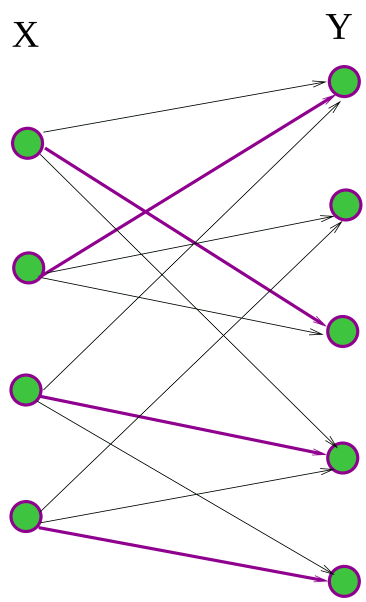
Network Problems using IP

- **Assignment Problem.**
- **minimize** $\sum c_{ij} x_{ij}$
- **subject to**

$$\sum_{j:(i,j)\in E} x_{ij} = 1 \quad \textbf{for all } i \in X$$

$$\sum_{j:(j,i)\in E} x_{ji} = 1 \quad \textbf{for all } i \in Y$$

$$x_{ij} \in \{0,1\} \quad \textbf{for all } (i,j) \in E$$



LP Relaxation of IP

- **An IP formulation with integrality constraint removed is called the LP relaxation.**
- **If IP is max cx, with optimal solution z_I, and z_L is the optimal solution of relaxed LP, then**

$$z_I \leq z_L$$

- **It follows because IP feasible space is a subset of LP feasible space.**
- **When can one guarantee that $z_L = z_I$?**
- **Specifically, shortest path and assignment formulations require integer variables. But they have combinatorial poly-time algorithms.**

Total Unimodularity

- **Consider an IP: max cx s.t. Ax $\leq$ b, and x integer.**
- **The constraint matrix A is called Totally Unimodular (TUM) if det(B) is in $\{0, 1, -1\}$ for all 2×2 submatrices B of A.**
- **E.g. if each col of A has exactly one 1, one -1, and all other zero entries, then A is TUM.**
- **If A is the incidence matrix of a graph, then A is TUM.**
- **Thus, our network problems (shortest path, maxflow, assignment) can be solved through LP relaxation.**