1. (20 pts) Modern Furniture Inc. produces two types of wooden chairs. Manufacturing chair A requires 2 hours of assembly time and 4 hours of finishing time. Chair B requires 3 hours to assemble and 3 hours to finish. Modern estimates that next week 72 hours will be available for assembly and 108 hours for the finishing operations. The unit profits for Chairs A and B are $10 and $9, respectively. If it is estimated that the maximum demand for Chair B is 16, what is the optimal product mix? Formulate a linear programming model and solve it using the graphical method.

2. (20 pts) Consider the following LP:

\[
\begin{align*}
\text{minimize} & \quad 47x_1 + 93x_2 + 17x_3 - 93x_4 \\
& -x_1 - 6x_2 + x_3 + 3x_4 \leq -3 \\
& -x_1 - 2x_2 + 7x_3 + x_4 \leq 5 \\
& 3x_2 - 10x_3 - x_4 \leq -8 \\
& -6x_1 - 11x_2 - 2x_3 + 12x_4 \leq -7 \\
& x_1 + 6x_2 - x_3 - 3x_4 \leq 4 \\
& x_1, x_2, x_3, x_4 \geq 0 
\end{align*}
\]

Prove, without using LP solver, that \((1, 1, 1, 1)\) is the optimal solution for this LP.

3. (30 pts) You are given a set \(A = \{a_1, a_2, \ldots, a_n\}\), where each item \(a_i\) has a non-negative weight \(w_i\), and a collection \(B_1, B_2, \ldots, B_m\) of subsets of \(A\). The goal is to choose a minimum weight subset of items \(H \subset A\) such that \(H \cap B_j \neq \emptyset\) for all \(j = 1, 2, \ldots, m\). Give a polynomial time algorithm for computing a set \(H\) with this property of weight at most \(b\) times the optimal where \(b = \max|B_j|\), that is, \(b\) is the maximum cardinality of any \(B_j\).

4. (10 pts) Show that if \(f\) is submodular, then

\[
\begin{align*}
f(A + e_1) + f(A + e_2) & \geq f(A) + f(A + e_1 + e_2) 
\end{align*}
\]

for \(\forall A \subset S\) and distinct \(e_1, e_2 \in S \setminus A\).