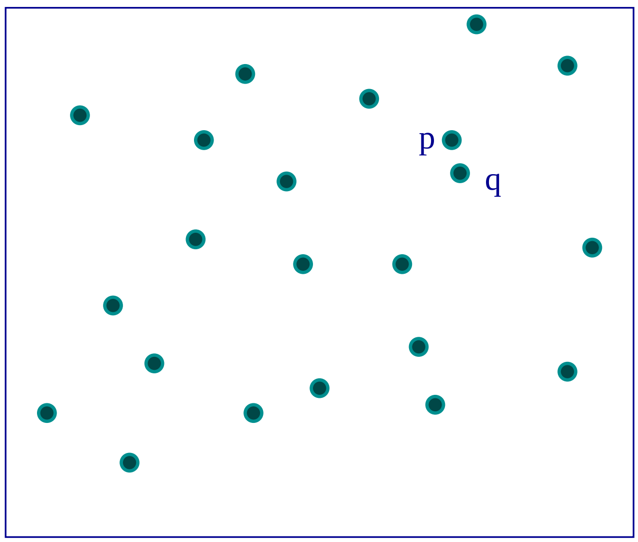
Closest Pair Problem

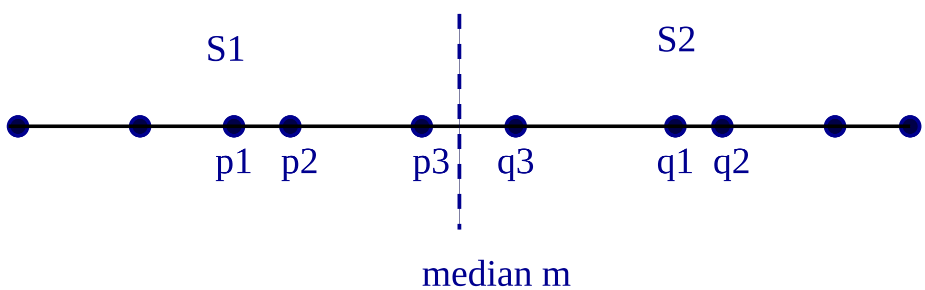
- **Given n points in d-dimensions, find two whose mutual distance is smallest.**
- **Fundamental problem in many applications as well as a key step in many algorithms.**



- **A naive algorithm takes $O(dn^2)$ time.**
- **Element uniqueness reduces to Closest Pair, so $\Omega(n \log n)$ lower bound.**
- **We will develop a divide-and-conquer based $O(n \log n)$ algorithm; dimension d assumed constant.**

1-Dimension Problem

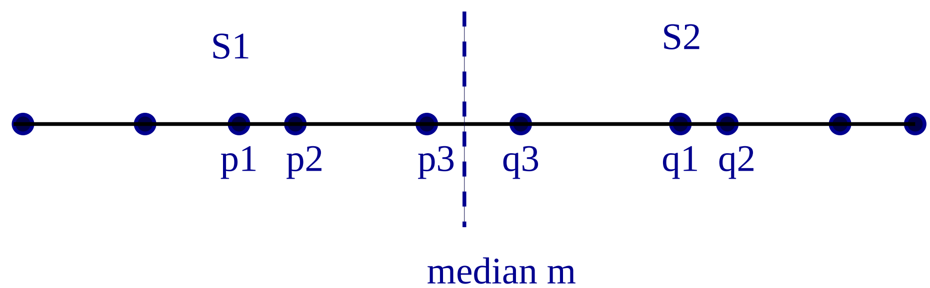
- **1D problem can be solved in $O(n \log n)$ via sorting.**
- **Sorting, however, does not generalize to higher dimensions. So, let's develop a divide-and-conquer for 1D.**
- **Divide the points S into two sets S_1, S_2 by some x-coordinate so that $p < q$ for all $p \in S_1$ and $q \in S_2$.**
- **Recursively compute closest pair (p_1, p_2) in S_1 and (q_1, q_2) in S_2.**



- **Let δ be the smallest separation found so far:**

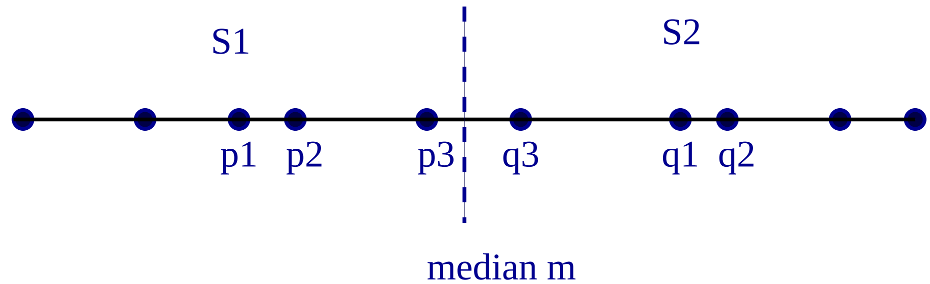
$$\delta = \min(|p_2 - p_1|,\ |q_2 - q_1|)$$

1D Divide & Conquer



- The closest pair is $\{p_1, p_2\}$, or $\{q_1, q_2\}$, or some $\{p_3, q_3\}$ where $p_3 \in S_1$ and $q_3 \in S_2$.
- **Key Observation:** If m is the dividing coordinate, then p_3, q_3 must be within δ of m.
- In 1D, p_3 must be the rightmost point of S_1 and q_3 the leftmost point of S_2, but these notions do not generalize to higher dimensions.
- How many points of S_1 can lie in the interval $(m - \delta, m]$?
- By definition of δ, at most one. Same holds for S_2.

1D Divide & Conquer



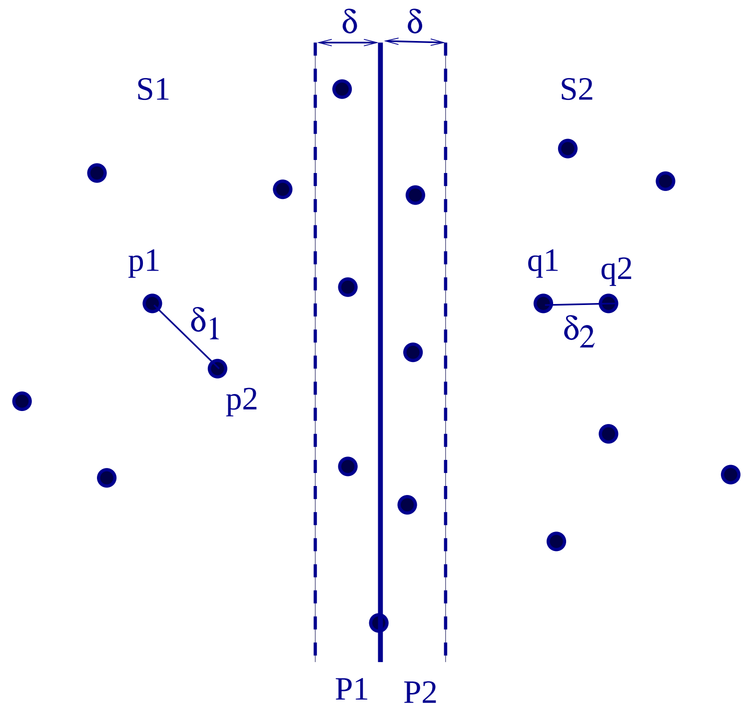
- **Closest-Pair** (S).
- **If** $|S| = 1$, **output** $\delta = \infty$.
  If $|S| = 2$, **output** $\delta = |p_2 - p_1|$.
  Otherwise, do the following steps:

  1. **Let** $m = median(S)$.
  2. **Divide** S **into** S_1, S_2 **at** m.
  3. $\delta_1 =$ **Closest-Pair**(S_1).
  4. $\delta_2 =$ **Closest-Pair**(S_2).
  5. δ_{12} **is minimum distance across the cut.**
  6. **Return** $\delta = \min(\delta_1, \delta_2, \delta_{12})$.

- **Recurrence is** $T(n) = 2T(n/2) + O(n)$, **which solves to** $T(n) = O(n \log n)$.

2-D Closest Pair

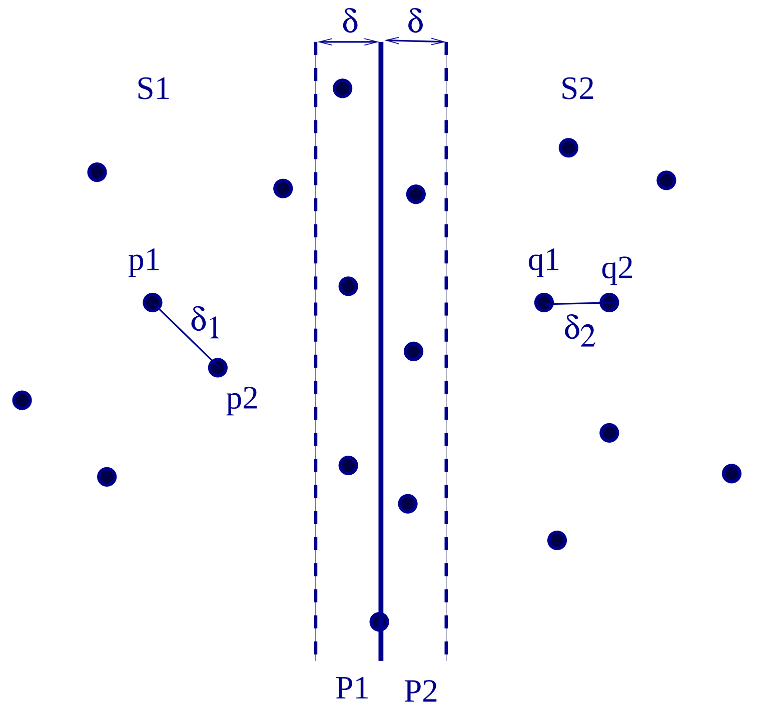
- **We partition S into S_1, S_2 by vertical line ℓ defined by median x-coordinate in S.**
- **Recursively compute closest pair distances δ_1 and δ_2. Set $\delta = \min(\delta_1, \delta_2)$.**
- **Now compute the closest pair with one point each in S_1 and S_2.**



- **In each candidate pair (p, q), where $p \in S_1$ and $q \in S_2$, the points p, q must both lie within δ of ℓ.**

2-D Closest Pair

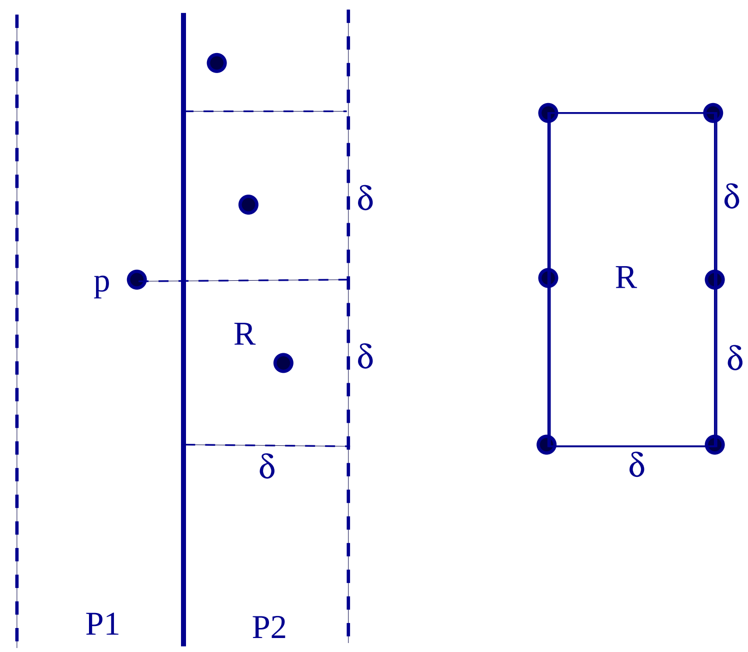
- **At this point, complications arise, which weren't present in 1D. It's entirely possible that all $n/2$ points of S_1 (and S_2) lie within δ of ℓ.**



- **Naively, this would require $n^2/4$ calculations.**
- **We show that points in P_1, P_2 (δ strip around ℓ) have a special structure, and solve the conquer step faster.**

Conquer Step

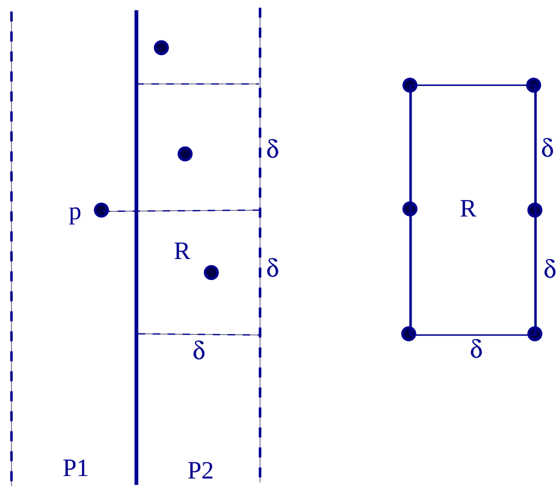
- **Consider a point $p \in S_1$. All points of S_2 within distance δ of p must lie in a $\delta \times 2\delta$ rectangle R.**



- **How many points can be inside R if each pair is at least δ apart?**
- **In 2D, this number is at most 6!**
- **So, we only need to perform $6 \times n/2$ distance comparisons!**
- **We don't have an $O(n \log n)$ time algorithm yet. Why?**

Conquer Step Pairs

- **In order to determine at most 6 potential mates of p, project p and all points of P_2 onto line ℓ.**

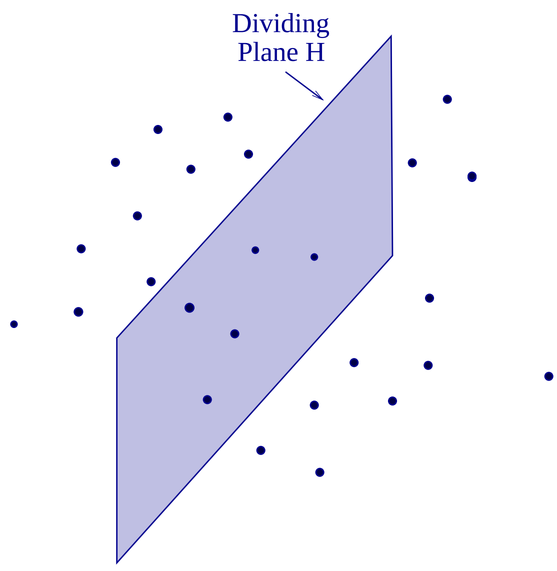


- **Pick out points whose projection is within δ of p; at most six.**
- **We can do this for all p, by walking sorted lists of P_1 and P_2, in total $O(n)$ time.**
- **The sorted lists for P_1, P_2 can be obtained from pre-sorting of S_1, S_2.**
- **Final recurrence is $T(n) = 2T(n/2) + O(n)$, which solves to $T(n) = O(n \log n)$.**

d-Dimensional Closest Pair

- Two key features of the divide and conquer strategy are these:

1. The step where subproblems are combined takes place in one lower dimension.
2. The subproblems in the combine step satisfy a sparsity condition.
3. **Sparsity Condition:** Any cube with side length 2δ contains $O(1)$ points of S.
4. Note that the original problem does not necessarily have this condition.



The Sparse Problem

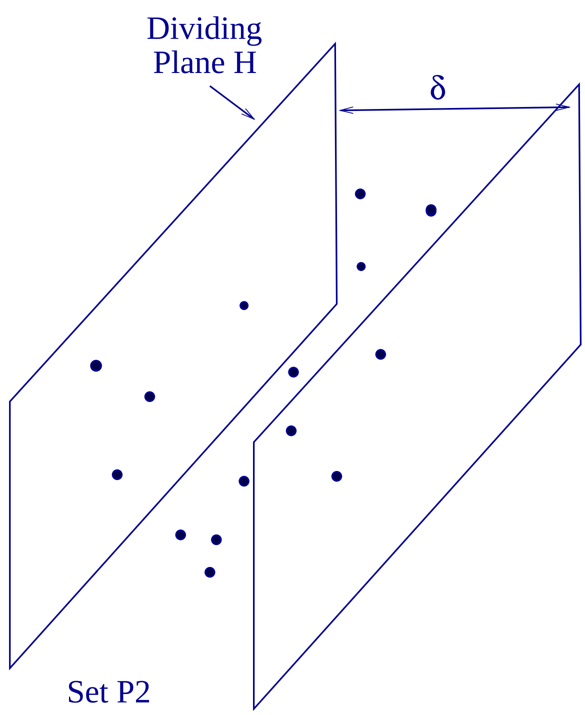
- **Given n points with δ-sparsity condition, find all pairs within distance $\leq \delta$.**
- **Divide the set into S_1, S_2 by a median place H. Recursively solve the problem in two halves.**
- **Project all points lying within δ thick slab around H onto H. Call this set S'.**
- **S' inherits the δ-sparsity condition. Why?.**
- **Recursively solve the problem for S' in $d-1$ space.**
- **The algorithms satisfies the recurrence**

$$U(n,d) = 2U(n/2,d) + U(n,d-1) + O(n).$$

which solves to $U(n,d) = O(n(\log n)^{d-1})$.

Getting Sparsity

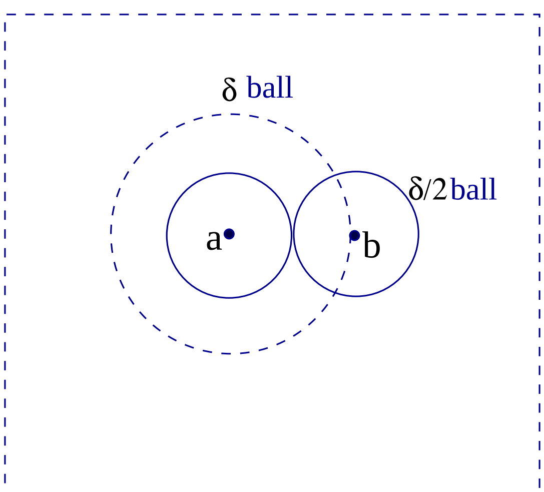
- Recall that divide and conquer algorithm solves the left and right half problems recursively.
- The sparsity holds for the merge problem, which concerns points within δ thick slab around H.



- If S is a set where inter-point distance is at least δ, then the δ-cube centered at p contains at most a constant number of points of S, depending on d.

Proof of Sparsity

- **Let C be the δ-cube centered at p. Let L be the set of points in C.**
- **Imagine placing a ball of radius $\delta/2$ around each point of L.**
- **No two balls can intersect. Why?**
- **The volume of cube C is $(2\delta)^d$.**
- **The volume of each ball is $\frac{1}{c_d}(\delta/2)^d$, for a constant c_d.**
- **Thus, the maximum number of balls, or points, is at most $c_d 4^d$, which is $O(1)$.**



Closest Pair Algorithm

- **Divide the input S into S_1, S_2 by the median hyperplane normal to some axis.**
- **Recursively compute δ_1, δ_2 for S_1, S_2. Set $\delta = \min(\delta_1, \delta_2)$.**
- **Let S' be the set of points that are within δ of H, projected onto H.**
- **Use the δ-sparsity condition to recursively examine all pairs in S'—there are only $O(n)$ pairs.**
- **The recurrence for the final algorithm is:**

$$\begin{aligned} T(n,d) &= 2T(n/2,d) + U(n,d-1) + O(n) \\ &= 2T(n/2,d) + O(n(\log n)^{d-2}) + O(n) \\ &= O(n(\log n)^{d-1}). \end{aligned}$$

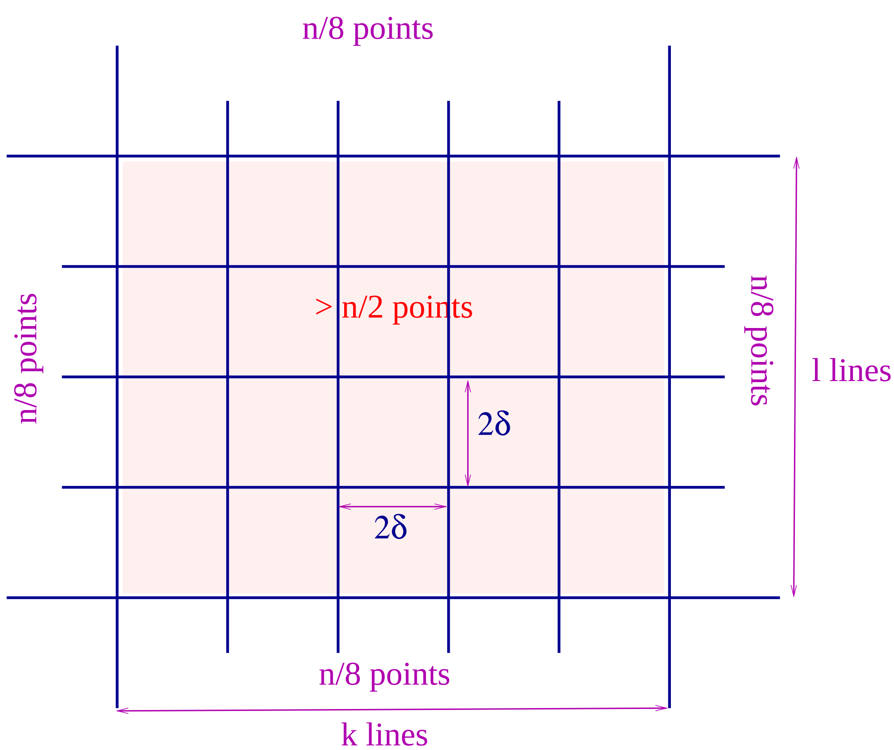
Improving the Algorithm

- If we could show that the problem size in the conquer step is $m \leq n/(\log n)^{d-2}$, then $U(m, d-1) = O(m(\log m)^{d-2}) = O(n)$.
- This would give final recurrence $T(n, d) = 2T(n/2, d) + O(n) + O(n)$, which solves to $O(n \log n)$.
- **Theorem:** Given a set S with δ-sparsity, there exists a hyperplane H normal to some axis such that

  1. $|S_1|, |S_2| \geq n/4d$.
  2. Number of points within δ of H is $O(\frac{n}{(\log n)^{d-2}})$.
  3. H can be found in $O(n)$ time.

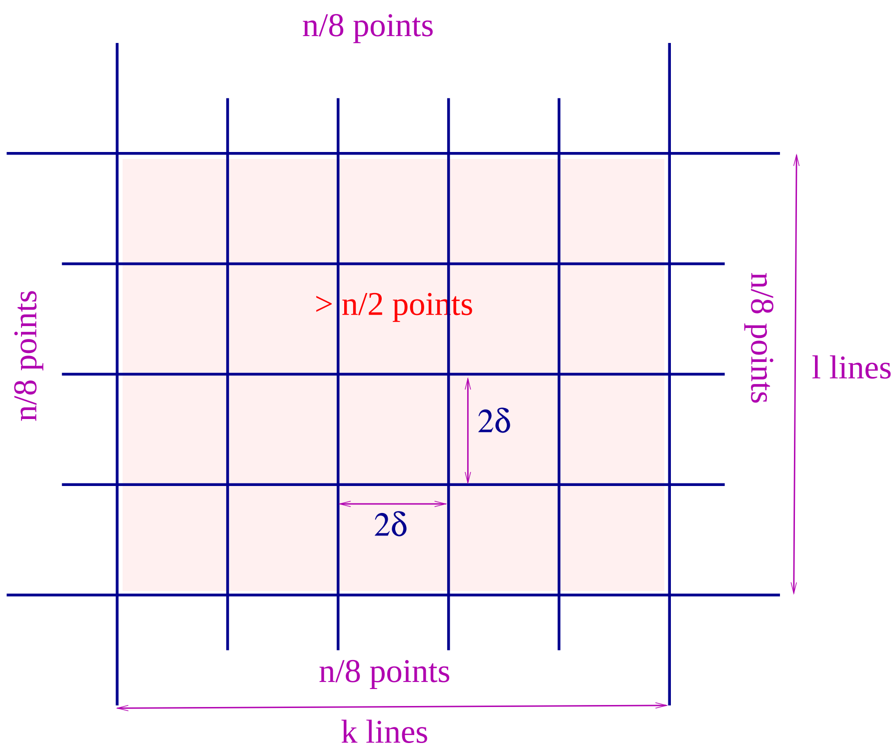
Sparse Hyperplane

- **We prove the theorem for 2D. Show there is a line with $\alpha\sqrt{n}$ points within δ of it, for some constant α.**
- **For contradiction, assume no such line exists.**
- **Partition the plane by placing vertical lines at distance 2δ from each other, where $n/8$ points to the left of leftmost line, and right of rightmost line.**



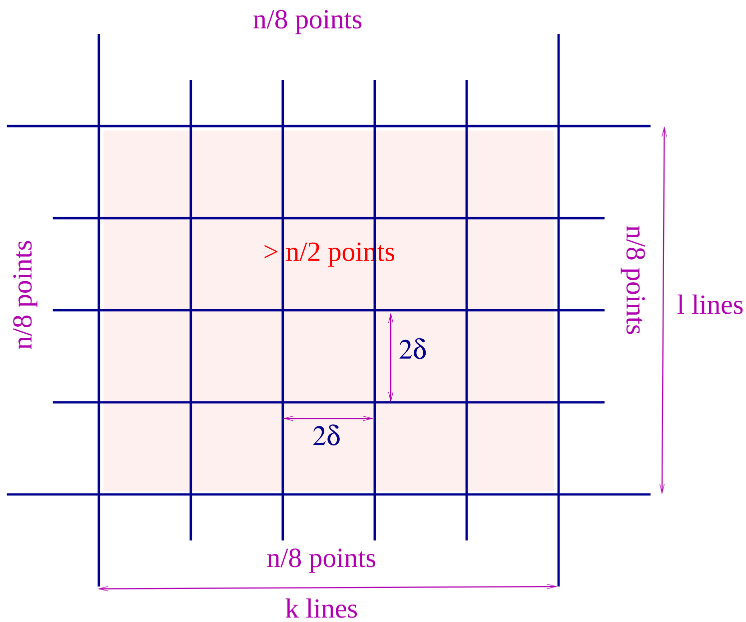
Sparse Hyperplane

- **If there are k slabs, we have $k\alpha\sqrt{n} \leq 3n/4$, which gives $k \leq \frac{3}{4\alpha}\sqrt{n}$.**



- **Similarly, if there is no horizontal line with desired properties, we get $l \leq \frac{3}{4\alpha}\sqrt{n}$.**
- **By sparsity, number of points in any 2δ cell is some constant c.**

Sparse Hyperplane



- This gives that the num. of points inside all the slabs is at most ckl, which is at most $\left(\frac{3}{4\alpha}\right)^2 cn$.
- Since there are $\geq n/2$ points inside the slabs, this is a contradiction if we choose $\alpha \geq \frac{\sqrt{18c}}{4}$.
- So, one of these k vertical of l horizontal lines must satisfy the desired properties.
- Since we know δ, we can check these $k + l$ lines and choose the correct one in $O(n)$ time.

Optimal Algorithm

- **Actually we can start the algorithm with such a hyperplane.**
- **The divide and conquer algorithm now satisfies the recurrence** $T(n,d) = 2T(n/2,d) + U(m,d-1) + O(n)$**.**
- **By new sparsity claim,** $m \leq n/(\log n)^{d-2}$**, and so** $U(m,d-1) = O(m(\log m)^{d-2}) = O(n)$**.**
- **Thus,** $T(n,d) = 2T(n/2,d) + O(n) + O(n)$**, which solves to** $O(n \log n)$**.**
- **Solves the Closest Pair problem in fixed** d **in optimal** $O(n \log n)$ **time.**