Theoretical Computer Science 84 (1991) 127-150
Elsevier

Shortest paths without a map

Christos H. Papadimitriou*
Department of Computer Science and Engineering, University of California, San Diego, USA

Mihalis Yannakakis
AT&T Bell Laboratories

Abstract

Papadimitriou, C.H. and M. Yannakakis, Shortest paths without a map, Theoretical Computer Science 84 (1991) 127-150.

We study several versions of the shortest-path problem when the map is not known in advance, but is specified dynamically. We are seeking dynamic decision rules that optimize the worst-case ratio of the distance covered to the length of the (statically) optimal path. We describe optimal decision rules for two cases: layered graphs of width two, and two-dimensional scenes with unit square obstacles. The optimal rules turn out to be intuitive, common-sense heuristics. For slightly more general graphs and scenes, we show that no bounded ratio is possible. We also show that the computational problem of devising a strategy that achieves a given worst-case ratio to the optimum path in a graph with unknown parameters is a universal two-person game, and thus PSPACE-complete, whereas optimizing the expected ratio is #P-hard.

1. Introduction

Finding shortest paths is one of the most well-looked at problems in Computer Science and Operations Research (see, for example, [7 16], and the classical survey by Dreyfus [4]). More recently, several versions of the shortest-path problem in a geometric setting have been considered (see, for example, [9, 10, 3, 13, 11, 12, 15] for a survey with over thirty references). Such variants are motivated by planning the motion of a robot in a scene sprinkled with obstacles, in which the metric is determined by the geometry of the obstacles.

It is sometimes natural to assume, both in the graph-theoretic and the geometric contexts, that the planner initially has incomplete information about the graph or scene, and such information is acquired in a dynamic manner, as the search for a good path evolves (e.g., by acquisition of sensory information in the geometric case, or by timed acquisition of the parameters, when the layer structure of a graph models time). What are appropriate search strategies in such a situation? And what are the right measures for evaluating such strategies?

* Research supported by the National Science Foundation.

Besides its inherent interest as a natural extension of a classical problem, this body of problems serves as an important paradigm in decision-making under incomplete information. Since it involves the design and evaluation of search heuristics, the techniques developed will add to the scarce rigorous methodological arsenal of Artificial Intelligence. As it turns out, the heuristics developed in this paper (and shown to be optimal in two important cases) have the flavor of nontrivial, yet natural and common-sense approaches to the problem.

To acquaint the reader with the kind of problems studied, consider the three examples in Figs 1–3. In Fig. 1 we have a layered graph of width two. Shortest-path problems for such graphs model dynamic decisions, as layers from left to right may model stages of the decision-making process, that is, time. Such problems can be solved by specialized *dynamic programming* techniques (in fact, these were the archetypical applications of this method). Imagine, however, that the graph is given to us *one stage at a time.* In the beginning, we only know the part shown in Fig. 1b. A rational searcher would probably try the lower choice. When the unfortunate information of the next stage (Fig. 1c) is revealed, should the searcher persist on this path? (He/she always has the choice of following edges backwards, thus switching paths, but the distance traversed this way is counted in the total score.) Obviously, there is no way to guarantee that the searcher always finds the shortest path (shortest in the static sense, as if the graph were known beforehand). *We assume that the goal of the searcher is to devise a strategy, so that the total distance traversed has the best possible ratio to the shortest path.* This is a rather familiar notion of performance from another area of research, namely on-line algorithms for assigning "servers" to "requests," a topic that recently has attracted wide attention (see, e.g., [18, 2, 8]). Although our problems come from a completely

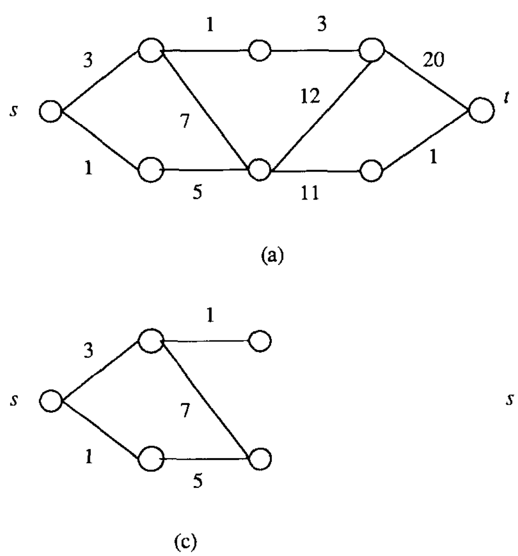


Fig. 1. Searching layered graphs of width two.

different application area (i.e., Robot Motion), and are typically graph-theoretic or geometric in nature, our work can also be seen as a contribution to on-line algorithms.

For example, a reasonable strategy for the problem of layered graphs of width two could be, informally: "Take the shortest edge out of the current node, unless there is a path to the other node that is less than half this one." Thus, in the example of Fig. 1, the searcher should persist in the second stage, but should switch in the next (Fig. 1d). We show (Theorem 2.4) that a variant of this strategy is optimal, and achieves a worst-case ratio of distance travelled to shortest path equal to 9, the best possible.

In Fig. 2, the searcher must traverse an unknown obstacle course and reach a goal point, again seeking to optimize the ratio of distance travelled to shortest path. The scene is not known a priori, but the perimeter of the obstacles becomes known to the searcher as he/she sees its various parts. What are reasonable strategies in this regime? One possible strategy, familiar to those who walk long distances among irregular city blocks, would be this: "When faced with a block, turn the nearest corner." We shall show that neither this nor any other strategy can achieve a constant ratio, even if the obstacles are nonintersecting rectangles with sides parallel to the axes. However, if the obstacles are *unit squares* (or squares of bounded size), an intuitive variant of the nearest-corner heuristic is guaranteed to be asymptotically at most 50% above the shortest path, *and this is optimal* (Theorems 3.2 and 3.7). For squares of arbitrary size (but with sides parallel to the axes) we have another heuristic that achieves a ratio $\sqrt{26}/3 \approx 1.7$.

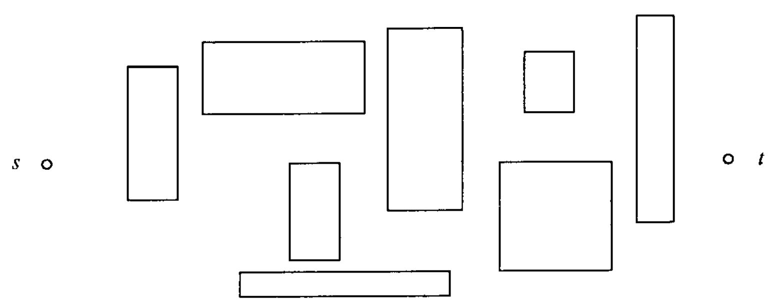


Fig. 2. Searching an obstacle scene.

We can also study such a situation as a computational problem. Consider Fig. 3. The road map is now known, but the roads with question marks may be unsuitable for travel (say, due to snowfall), an eventuality that is revealed to us only when an adjacent node is reached. What is the computational complexity of devising a travel strategy which guarantees a given ratio to the shortest (feasible) path? We call this the CANADIAN TRAVELLER'S PROBLEM. It is, in fact, the specification of a *two-person game*, between a searcher and a malicious adversary, who sets the weather conditions so as to maximize the ratio. As it turns out, *it is a* PSPACE-*complete problem* (Theorem 4.1). Simple geometric situations are shown to be equally intractable.

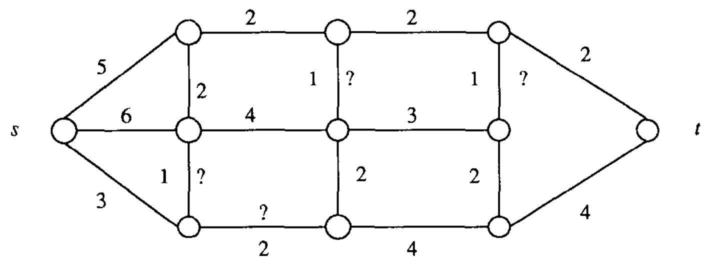


Fig. 3. The Canadian Traveller's Problem.

Variants in which we are given probabilities of unavailability of the edges (or presence of obstacles in the geometric case) and we are asked to minimize the expected ratio to the optimum are also intractable (#P-hard, and solvable in polynomial space).

2. Layered graphs

A layered graph is a graph in which the nodes are partitioned into layers $L_1, \ldots, L_n$, and all edges are between adjacent layers. We consider layered graphs of width two, that is, $|L_i| \leq 2$ for all i. We assume that the edges between L_i and L_{i+1}, and their lengths, become known only when a node in L_i is reached; the number of layers is also unknown. Edges can be traversed backwards, and the lengths are nonnegative integers. The first and last layers have one node.

We are interested in determining the strategy for searching such graphs that achieves the best possible worst-case ratio of the length of the path traversed, divided by the optimum path. This coincides with the value of the following game played on layered graphs of width two: The game is between two players, the searcher and the adversary. At the beginning of stage i a layered graph of width two and i layers has been revealed. The searcher is at one of the two nodes of the last layer. Then the adversary moves, and describes a new layer with at most two nodes, the edges connecting these new nodes with nodes of the previous layer, and their costs. The searcher must move from the current node to one of the nodes in the new layer, using edges of the graph. Also, the adversary may declare the new layer to be the last one. In this case the ratio of the total distance covered by the searcher divided by shortest path from the first layer to the last layer is computed. The searcher pays the adversary this amount.

Consider the following family of standard strategies for the adversary: At the first play add two nodes, connected with edges of length 1 to the start node. At each subsequent play except for the last, add two new nodes, each connected with a node in the previous layer by an edge. The edge out of the node occupied by the searcher has length 1, the other has length 0. A graph created by a standard strategy is depicted in Fig. 4. The positions of the searcher are marked by a $*$.

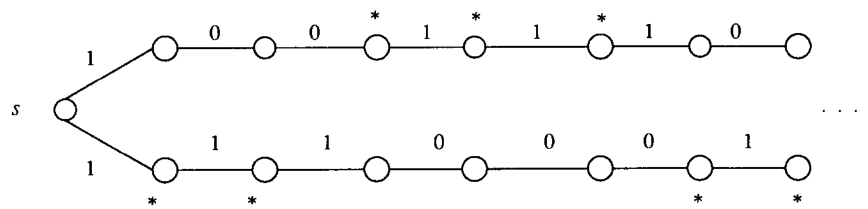


Fig. 4. The adversary's standard strategy.

Lemma 2.1. *For any strategy for the adversary, there is a standard strategy that achieves at least the same ratio.*

Proof. We have to show that if the searcher can achieve a given ratio against standard strategies, then he can achieve the same ratio against all adversary strategies. We will give the proof in three steps. Consider an arbitrary strategy τ of the adversary. At stage k the adversary presents the lengths of the edges from the nodes u_1, u_2 of the kth layer to the nodes v_1, v_2 of the next layer. Since the searcher can backtrack, we can assume without loss of generality that the length of each edge (u_i, v_j) does not exceed the length of the shortest path from u_i to v_j through the portion of the graph seen so far. The shortest paths from the start node s to the nodes u_1, u_2 of the kth layer (with ties broken arbitrarily) form a rooted tree T_k with root s and the two leaves u_1, u_2. Similarly, the shortest paths from s to v_1, v_2 form a tree T_{k+1} which is an extension of T_k, i.e. it is obtained by hanging each one of v_1 and v_2 from one of the previous leaves.

Consider the strategy τ_1 of the adversary defined as follows: If node v_i hangs from u_j in the tree T_{k+1}, then the adversary includes the edge (v_i, u_j) with the same length as in τ, but does not include the edge to v_i from the other node u_l of the previous layer (i.e. $l \neq j$), or equivalently, the other edge (u_l, v_i) is given the length of the shortest weighted path from u_l to v_i through the tree T_{k+1}. It should be intuitively clear that the graph can only become harder for the searcher. This can be easily formalized. Clearly, the distances from the start node to the nodes of each layer are identical in the two strategies, and the same is true of the trees T_k. Given any strategy of the searcher in response to the strategy τ_1 of the adversary, if the searcher makes the exact same choices in response to τ, then he achieves at least as good a ratio against τ as against τ_1. Consequently, we may restrict our attention to strategies of the adversary with the property that every node v_i of each layer has an edge only to one node of the preceding layer, and the length of the other edge incident to v_i is the distance implied by the triangle inequality. Let us call these strategies type 1.

Let type 2 be the subset of adversary strategies τ_2 in which the two nodes of each layer are hung from distinct nodes of the previous layer, and let r be the best possible ratio achievable by a type 2 strategy. Thus, there is a strategy σ_2 for the searcher which can guarantee ratio r against a type 2 strategy. Then we claim that there is a searcher strategy σ_1 that guarantees ratio r also against any adversary strategy of type 1. The searcher operates in rounds as follows. As long as the tree of the shortest paths consists of two disjoint paths from the start node to the nodes of the current level, the searcher follows the choices of σ_2. Suppose that the adversary reveals two nodes v_1, v_2 in the next layer, both hanging from the same node, say u_1, of the current layer (the other possibility where the next layer has only one node is easy). The searcher first goes to u_1, if he is not already there, and regards the current round as being finished. The distance traversed until reaching u_1 is within a factor of r of the shortest path from the start node to u_1. The reason is that the adversary could have made instead the next layer be the last one with the goal node hanging from u_1 with an edge of length 0. After the searcher reaches u_1, he erases all the preceding layers, begins a new round with start node u_1, lets the nodes v_1, v_2 be the first layer, and starts again simulating σ_2 on the new graph. Therefore, type 2 strategies for the adversary are as powerful as general strategies.

The difference between a type 2 and a standard strategy is that in a standard strategy the adversary reveals the information one unit at a time, in an effort to entice the searcher to perform useless work. It should be intuitively clear that this makes things only harder for the searcher. This can be easily formalized to prove that standard strategies are as powerful as type 2, to complete the proof. □

Lemma 2.1 gives rise to the following game, called SEQ: Player I defines an infinite sequence of positive integers $\langle a_1, a_2, \ldots \rangle$ such that (i) $a_1 = a_2 = 0$; and (ii) the even and the odd subsequences are increasing and unbounded. Then Player II picks a positive integer $n > 2$. Player I pays Player II $(2\sum_{i=1}^{n} a_i + a_{n-1})/a_{n-1}$.

Lemma 2.2. *The value of* SEQ *equals the optimum ratio for traversing layered graphs of width two.*

Proof. The standard adversary strategy defines two paths. The searcher's strategy defines the path lengths at which a path is abandoned for the other. These lengths are precisely the a_i's in SEQ. The adversary's strategy is reduced to choosing the end of the game. □

It turns out that the value of SEQ has been computed by researchers working on another, more geometric, problem: We have a two-way infinite line and a searcher starting from the origin. The searcher can move in unit steps, and wants to find a goal that lies at some unknown distance d. The objective of the searcher is to minimize the ratio of the distance traversed to the true distance d.

Lemma 2.3 (Baeza-Yates et al. [1]). *The value of* SEQ *is* 9. *It is asymptotically achieved by taking the* a_i's *to be powers of* 2.

Consider thus the following intuitive strategy for the searcher: "If the shortest paths from the start node to the nodes of the next layer go through the same node u of the current layer, then go first to this node u, erase all the preceding layers and regard u as the new start node. In either case, follow the shortest edge out of the current node to a node of the next layer, unless there is a path from the start node to the other node in the next layer, whose length is less than half the length of the (shortest) path from the start to the considered node. In the latter case, follow the shortest path to the other node of the next layer." From Lemmata 2.1, 2.2, and 2.3 we have the following:

Theorem 2.4. *The strategy above is optimal, and achieves a ratio of* 9, *the best possible.*

If we consider layered graphs with unbounded width w then it is easy to see that no fixed ratio is possible. A lower bound of w is trivial. Furthermore, [1] analysed the generalization of SEQ where we have w half-lines emanating from the origin, and a searcher at the origin who wants to locate a goal that lies on one of the lines, and he wants again to minimize the ratio of the distance traversed to the true distance. They showed that the optimal search strategy is to rotate among the half-lines travelling distances that form a geometric series with ratio $\rho = w/(w-1)$. The optimal ratio to the shortest path, which is achieved by this strategy, is

$$r_w = \frac{2w^w}{(w-1)^{w-1}+1};$$

its limit, as w grows, is $2ew+1$. It is easy to see that this problem is a special case of searching layered graphs of width w, when the adversary uses a standard strategy; thus r_w is a lower bound for the best ratio achievable.

If every stage of the layered graph is a perfect matching, i.e., consists of exactly one edge from each node of the current layer to a distinct node of the next layer, then the layered graph consists of w paths out of the source with no cross edges, and the situation is clearly similar to that of searching for a goal along w half-lines. In this special case the ratio r_w is also an upper bound. One strategy that achieves ratio r_w is the following. Initially, set a counter c to 0 and take the shortest edge out of the source. In each stage, follow the edge out of the current node continuing on the current path unless the length of the path is about to exceed ρ^c, where $\rho = w/(w-1)$. In the latter case, go to the node of the next layer that is closest to the source, say it is at distance d from the source, and reset c to the maximum of $c+1$ and $\lfloor \log_\rho d \rfloor + 1$.

Although it may appear that the adversary should not introduce cross edges because they would only help the searcher, it turns out that this is not the case. That is, when w is three or more, the best strategy of the adversary is not standard anymore, and the above lower bound r_w is strict for general layered graphs.

3. Obstacle courses

Suppose that a searcher must traverse a two-dimensional scene with impenetrable obstacles from a start position to a goal position (recall Fig. 2). Our assumptions are the following: (i) The obstacles are nonintersecting rectangles with sides parallel to the axes. (ii) The location of the goal is known to the searcher. (iii) The obstacles become known to the searcher as they come within the searcher's line of vision (that is, the searcher at any moment knows all parts of the perimeter of the obstacles that can be joined with an obstacle-free straight line to a past position of the searcher). Notice that our algorithms use far weaker information (that is, full knowledge of the side of an obstacle is acquired only when the side is reached by the searcher), whereas our lower bounds are valid even when full visual information is assumed. (iv) Neither the start position nor the goal are within an obstacle. Again, we wish to find strategies which achieve a bounded ratio with the shortest obstacle-free path. Unfortunately, the following result holds:

Proposition 3.1. *There is no strategy that achieves a bounded ratio.*

Proof. Consider Fig. 5. The goal is a horizontal distance n from the start, and all obstacles are rectangles of very small width, length n, and placed at integer x coordinates. Initially, the searcher may avoid the first obstacle by going to one of its corners (up or down, down in the Figure). Then an obstacle is placed in front of him/her. The searcher goes either up or down, to meet the next obstacle, and so on. There are no other obstacles. The total length traversed by the searcher is $\Omega(n^2)$. However, there is a y-coordinate smaller than $n^{3/2}$ that meets $O(\sqrt{n})$ obstacles (since there are n obstacles in total each of length n), and thus the shortest path is $O(n^{3/2})$.

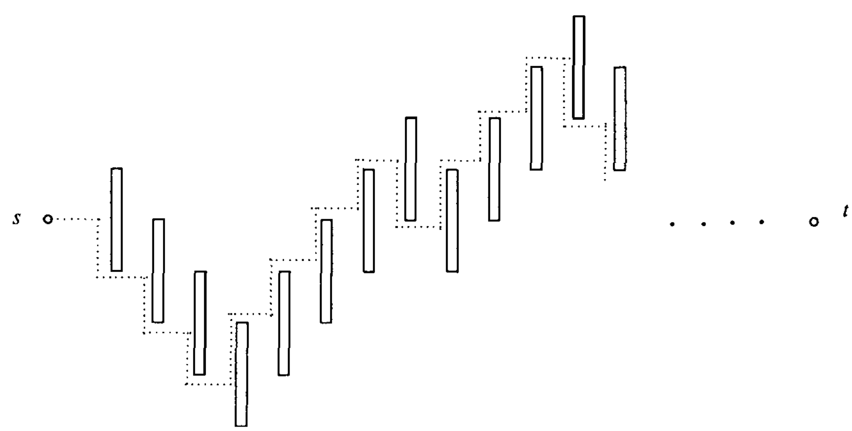


Fig. 5. A difficult scene.

This is a valid lower bound if the searcher never goes back, or learns the scene only by tactile sensing, that is, an obstacle is sensed only when its perimeter is reached. However, under our assumption of visual information (assumption (iii) above) there are conceivable strategies under which the searcher goes back to take a good look at the scene, and then proceeds. We can make visual information useless by supplementing Fig. 5 with some more obstacles. In particular, think of the scene as one with one very long vertical obstacle (a wall) at each integer x-coordinate i, with two very thin horizontal openings at the boundaries of the obstacle with x-coordinate i in Fig. 5 and with thin openings very close to all the other integer y-coordinates that are in the empty space in the figure. By close, we mean that the holes are just slightly out of alignment, so that the only thing that the searcher can see through a hole is that there is another obstacle behind it at the next integer x-coordinate. It is easy to see that the shortest path has still approximately the same length, and visual information can now be of no help, so the lower bound holds. □

The problem with Fig. 5 is that the obstacles are arbitrarily thin. What if they have a bounded aspect ratio? Even when they are squares, we have the following:

Theorem 3.2. *There is no strategy that achieves a ratio better than $\frac{3}{2}$ in the case of square obstacles.*

Proof. The proof is the same as in Proposition 3.1, with unit squares replacing the long rectangles; only now the vertical motion can be guaranteed to be at least half the horizontal one (which is n). Finally, the shortest path is again along a y-coordinate which passes through at most $\sqrt{n}$ obstacles, and is thus of length $n+O(\sqrt{n})$. The lower bound follows. Visual information can be rendered useless by a similar (but more tedious) construction as in the Proposition. □

Notice that the lower bound of Theorem 3.2 is achieved by *identical unit* squares.

Can the ratio of $\frac{3}{2}$ be achieved by a strategy? For example, the intuitive strategy: "Proceed horizontally. When faced by an obstacle, go to its nearest corner (up or down)" may lead to a factor arbitrarily close to 2 (for example, if the nearest corner is always up, by very little).

A little more care is needed to improve on 2. We first notice that, if the line from the start to the goal forms an angle of 45° with the axes, a bound of $\sqrt{2}$ (better than $\frac{3}{2}$!) is possible: Go directly towards the goal, avoiding obstacles in the obvious way (Fig. 6). We can combine this observation with the nearest-corner heuristic as follows: "Act according to the nearest-corner heuristic, until a position is reached that forms an angle of 45° when joined by a line to the goal. From then on, act according to the 45° heuristic." (Fig. 7). We call this the *mixed heuristic.* An elementary calculation in Fig. 7(a) shows that this rule achieves a ratio of $\frac{5}{3}$ *when start and goal are in the same horizontal level.* When they are at different horizontal levels, at some angle ϕ with the horizontal line (Fig. 7(b)), the situation may be a little worse:

Theorem 3.3. *The mixed heuristic achieves a ratio of $\frac{1}{3}\sqrt{26}$.*

Proof. Consider Fig. 7(b). The worst-case distance traversed, in units of the distance from start to goal, is $\cos\phi+\sin\phi+2\cdot x$. We calculate x, by the law of sines of the

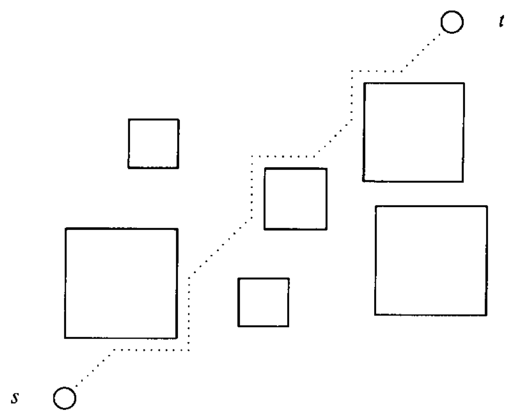
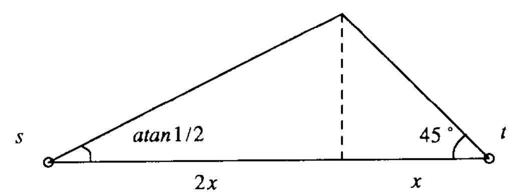
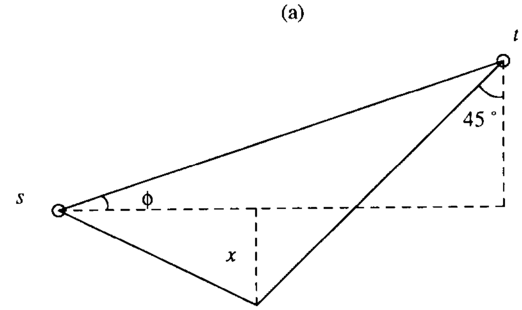


Fig. 6. The 45° heuristic.



(a)



(b)

Fig. 7. The mixed heuristic.

obtuse lower triangle, to be $\frac{1}{3}(\cos\phi - \sin\phi)$, and so the distance traversed is at most $\frac{1}{3}(5\cos\phi + \sin\phi)$. This expression is maximized when $\phi = \arctan\frac{1}{5}$, and the optimum is $\frac{1}{3}\sqrt{26} \approx 1.7$. $\square$

We can do better when the squares have size at most 1 (or more generally, bounded by a constant); recall that $\frac{3}{2}$ is a lower bound in the case of unit squares as well. Assume first for simplicity that the start and the goal are on the same horizontal level. To motivate the new heuristic, recall what is wrong with the nearest-corner heuristic: It may lead us away from the x-axis (that is, the line from start to goal). Intuitively, when we are away from the x-axis, the corner that is closest to the x-axis may be more attractive than the other, even though the other corner may be closer to the searcher's current position. The mixed heuristic can be explained in the light of this insight: When we come very close to the goal in horizontal distance (cross the 45° line), we start preferring the corner closest to the x-axis at all costs. (In fact, we can improve slightly on the constant $\frac{1}{3}\sqrt{26}$ by preferring the upper angle by an appropriate amount when the goal is above the horizontal level of the starting point.) Is there a more continuous, "smooth" way to bias our choice of corners? Our final heuristics do exactly this, and achieve asymptotically (as the distance from start to goal grows) the optimum ratio of $\frac{3}{2}$.

We will describe first a heuristic for the simpler case that the start and the goal are in the same horizontal line, all the squares have (exactly) unit size and have their sides parallel to the axes.

Define $\varepsilon = 1/\sqrt{n}$, where n is the distance from start to goal (measured in sides of the squares). When faced with an obstacle, the searcher has a *bias* β towards the corner which is closer to the x-axis. That is, we prefer the corner closer to the x-axis if it is less than $\frac{1}{2}+\beta$ away; otherwise, we choose the other corner. Initially $\beta = 0$, and every time we move to the other side of the x-axis, we reset β to 0. Every time we choose the corner away from the x-axis, β is increased by ε. Every time we choose the corner towards the x-axis although it is the further corner, the bias is decreased by ε; if the corner towards the x-axis is the closest corner, then we do not change β. We call this the *bias heuristic.* When we arrive at the same x-coordinate as the goal, then we go directly to it using for example the straightforward heuristic with ratio 2. Or we may switch to the 45° heuristic when we are at a 45° angle from the goal.

Note that at all times, the bias is an integer multiple of ε. It is an easy calculation to show that, in the bias heuristic: (i) At each step we are traversing a vertical distance which is at most β larger than $\frac{1}{2}$; (ii) When we do traverse a vertical distance that exceeds $\frac{1}{2}$ by the bias β, then in some previous step we must have traversed a distance at least $\beta - \varepsilon$ *less* than $\frac{1}{2}$, for a balance of ε. (iii) This balance of ε per square is insignificant when compared to the horizontal distance of n, since $n \cdot \varepsilon = \sqrt{n}$. (iv) At any given time, if $\beta = k\varepsilon$, then we are not more than $1 + \frac{k}{2}$ off from the x-axis because, on the one hand, every time we move away from the x-axis, we move by *at most* $\frac{1}{2}$ and increase the bias, while on the other hand, every time we decrease the bias we come closer to the x-axis by *at least* $\frac{1}{2}$. Since β is never larger than $\frac{1}{2}$, we never go further than $\frac{1}{2}/\varepsilon = \mathrm{O}(\sqrt{n})$ from the x-axis. Therefore, the total distance traversed to the goal is no more than $3n/2 + \mathrm{O}(\sqrt{n})$. Hence we have:

Proposition 3.4. *Suppose that all obstacles are unit squares with sides parallel to the axes, and that the goal is at the same horizontal distance with the start position, at a distance of n (measured in sides of the unit square). Then the bias heuristic achieves, as n grows, a ratio arbitrarily close to $\frac{3}{2}$, the best possible.*

We will relax now the assumption that the start and the goal are in the same horizontal line and that all the obstacles are identical unit squares, but allow them to be any squares of size at most 1. In order to make the exposition clearer, we will keep for now the assumption that the sides of the squares are parallel to the axes; at the end we will indicate how this restriction can be lifted.

If the goal is at an angle with respect to the start position, then things become somewhat more complicated. We can assume without loss of generality that the goal t is up and to the right of the start position s, and that the line from s to t forms an angle $\phi \leqslant 45°$ with the x-axis. Let d denote the direction from s to t. At any given time, let p denote the position of the searcher and let q be its projection on the s–t line. Our heuristic will have the property that p does not get very far off from the s–t line, never more than $O(\sqrt{n})$. Starting from s we will proceed in direction d avoiding the obstacles when they are in the way. When we avoid an obstacle, we would like the ratio of the distance travelled by p to the distance travelled by q in the direction d (i.e., progress towards the goal) to be bounded by $\frac{3}{2}+\varepsilon$. We cannot always guarantee this while staying close to the s–t line, but as in the previous case, we will achieve it in an amortized sense: if the ratio is larger than $\frac{3}{2}$ for some obstacle, then the excess work is cancelled by work saved from a previous obstacle.

Suppose that while travelling in direction d we hit an obstacle $ABCD$ at a point P, see Fig. 8. We either hit the side AD or AB. In the first case, we go around D, up the side DC part of the way until we meet the line from P parallel to d, at which point we resume our movement in the direction d. The ratio of the distance travelled by p to that covered by q is $\cos\phi+\sin\phi \leqslant \sqrt{2} \leqslant \frac{3}{2}$. We act similarly if P lies in the interval BE (see Fig. 8), that is, we go around B and then continue on the line passing through P in the direction d. Note that if $\phi = 45°$, then $E = A$, the above cases cover all the possibilities, and we simply have the 45° heuristic.

So we only need to consider the case that P lies in the interval AE. Let $y = |AP|$ (the length of the segment AP), let $\tau_1 = |PB| + |BC|$ be the distance travelled if we go from P up to B and then to C around the square, $\tau_2 = |PA| + |AD|$ the distance

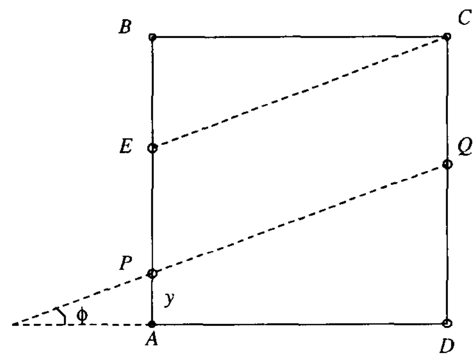


Fig. 8. Hitting a square at an angle.

if we go to A and then to D. Let π_1 be the projection of PC along the direction d, and π_2 the projection of PD along d.

Lemma 3.5 (Fejes Toth [5]). *The ratio τ_1/π_1 is a monotonically decreasing and the ratio τ_2/π_2 is a monotonically increasing function of the distance y of P from A. Furthermore, for every point P of the side AB, at least one of the two ratios is no more than $\frac{3}{2}$.*

The proof is not very difficult, see [5]. The Lemma tells us that going to at least one of the two corners C or D, guarantees a ratio of $\frac{3}{2}$, but the problem is we may move further away from the s–t line. Draw from P the line in direction d, and let Q be the point at which it intersects the side CD (see Fig. 8).

Lemma 3.6.

$$\frac{|QD|\tau_1+|QC|\tau_2}{|QD|\pi_1+|QC|\pi_2}\leqslant\frac{3}{2}.$$

Proof. Clearly, all the quantities scale with the size of the square. That is, the size of the square does not influence the Lemma, so let us assume we have a unit square. Let $a=\sin\phi$, $b=\cos\phi$. If $|AP|=y$, then $|QD|=y+\frac{a}{b}$ and $|QC|=1-y-\frac{a}{b}$. Also we can calculate, $\tau_1=2-y$, $\tau_2=1+y$, $\pi_1=a+b-ay$, and $\pi_2=b-ay$. The denominator in the ratio of the Lemma is $(a^2+b^2)/b=\frac{1}{b}$, and the numerator is $f(y)/b$, where $f(y)=(a+b)+2(b-a)y-2by^2$. Taking the derivative of f, we see that this function is increasing from $y=0$ to $(b-a)/2b$ and then is decreasing. Substituting $y=(b-a)/2b$ into f, the inequality of the Lemma, $f(y)\leqslant\frac{3}{2}$, reduces to $(b-1)(b-\frac{1}{2})\leqslant 0$, which is true, since $b=\cos\phi$ and $\phi\leqslant 45°$. □

Let us partition the interval AE of the side of a unit square into equal segments of length $\varepsilon=\frac{1}{\sqrt{n}}$ each, where n is the straight-line distance from s to t. For $i=0,1,\dots,$ the ith segment contains the points on AE with y between $i\varepsilon$ and $(i+1)\varepsilon$. Similarly, we partition the interval AE of a square of size $K\leqslant 1$ into corresponding segments of length εK. We label every segment "up" or "down" as follows: a segment is labelled up if its lowest point satisfies $\tau_1/\pi_1\leqslant\frac{3}{2}$, and is labelled down if the lowest point satisfies $\tau_2/\pi_2\leqslant\frac{3}{2}$. From Lemma 3.5; all segments have at least one label, and some may have both. We call the last segments *mixed*, and the segments with only one label *pure*.

Let P_i be the lowest point of the ith segment and let Q_i be the intersection of the line from P_i in direction d with the side CD. If the ith segment is pure, we define $\rho_i=|DQ_i|/CQ_i|$ if the segment is labelled only up, and $\rho_i=|CQ_i|/|DQ_i|$ if the segment is labelled only down.

The heuristic maintains a number, called a *balance*, with every pure segment. Initially all balances are 0. We already described how the searcher moves if he hits an obstacle in the intervals AD or EB. Suppose that he hits at a point P that lies on AE and belongs to the ith segment. Let us assume that the searcher is above the s–t line; the other case is similar. If the ith segment is labelled down, then the searcher moves to corner D and continues from there in direction d; the balance of the segment does not change. Suppose that the segment is not labelled down, i.e., is a pure segment labelled only up. Let $K \leq 1$ be the sidelength of the square. If the balance of the segment is at least $\rho_i K$ then the searcher subtracts $\rho_i K$ from the balance and moves to corner D as above. Otherwise, he adds K to the balance and moves up to corner C and resumes in direction d. We call this the *balance heuristic.*

As we shall prove, the searcher does not move more than $O(\sqrt{n})$ off the s–t line. When he comes to the same x- or y-coordinate as the goal (or at a 45° angle), then he can go directly towards the goal. This part of the travel is $O(\sqrt{n})$, thus negligible compared to the rest.

Theorem 3.7. *The balance heuristic achieves as n grows a ratio arbitrarily close to* $\frac{3}{2}$.

Proof. To prove the Theorem, we shall show that at all times (i) the distance of the searcher from the line s–t is $O(\sqrt{n})$, and (ii) the distance travelled by the point p representing the searcher is no more than $\frac{3}{2}+\varepsilon$ times the distance covered by its projection q on the s–t line.

Clearly, we only have to consider the obstacles that are hit in the interval AE. To simplify the arguments, we may assume that whenever the searcher hits an obstacle at the ith segment, he hits it at the lowest point P_i of the segment. The error introduced by this approximation is an $O(\varepsilon)$ factor of the side of the square, both in terms of the travel, and in terms of the distance from the s–t line. Thus, the total error over all the squares encountered is no more than $O(\sqrt{n})$.

To bound the distance travelled, let us consider the ratio of the travel of p to the travel of q when avoiding obstacles that are hit in the ith segment. If the segment has both labels, then clearly this ratio is no more than $\frac{3}{2}$. Let us assume that the segment is labelled up. When the searcher is below the s–t line, then whenever he hits the obstacle in the ith segment he travels up to avoid it, and again the ratio in this case is at most $\frac{3}{2}$. So we only have to consider the times the searcher is above the s–t line and hits the segment. Let k_i be the sum of the sides of the squares for which the searcher went to corner C and let l_i be the sum for the squares he went to corner D. Since the balance never becomes negative, we have $\rho_i l_i \leq k_i$. The corresponding total travel is $k_i\tau_1 + l_i\tau_2$, and the total distance covered by the projection q is $k_i\pi_1 + l_i\pi_2$, where the quantities τ_1 etc. are used here with respect to the lowest point P_i of the ith segment of a unit square. From Lemma 3.6 we have

$$\frac{\rho_i\tau_1+\tau_2}{\rho_i\pi_1+\pi_2} \leq \frac{3}{2},$$

and since the segment is labelled up, also $\tau_1/\pi_1 \leq \frac{3}{2}$. It follows that the contribution of the ith segment satisfies the $\frac{3}{2}$ ratio.

Let us consider now the vertical distance from the s–t line. Let us assume without loss of generality that the searcher is above the line. We only need to take into account the activity since the last time the s–t line was crossed. Clearly, the down segments only contributed in bringing the searcher closer to the line. Consider the ith segment, labelled up. Let b_i be the value of the corresponding balance. From the definition of the heuristic, the balance is always smaller than ρ_i+1. The balance was nonnegative when the searcher last crossed the s–t line. Therefore, the total vertical deviation above the s–t line that is due to the times that the searcher hit the ith segment is at most $b_i|CQ_i| \leq (\rho_i+1)|CQ_i| = 1$. Thus, the claim follows because there are (at most) $\sqrt{n}$ segments. □

We can extend the heuristic to the general case of squares with size at most one (or bounded size) that may not be aligned with the axes but can have arbitrary orientations. The basic idea is to discretize the slope of the squares. The direction d from the start s to the goal t hits the "hard" side of a square (side AB in Fig. 8) at an angle ϕ, which is between $-45°$ and $45°$. We divide the squares according to their slope ϕ into $n^{1/3}$ groups, so that $\cos\phi$ and $\sin\phi$ do not change more than $O(n^{-1/3})$ within each group. We partition the side of a square into $n^{1/3}$ segments now. From every group of squares we pick one representative, and again for each segment we pick an appropriate point P_i to represent it. If we assume that whenever we hit a square at a point P, then the square is the representative in its group, and P is the representative point in its segment, then the error introduced by this approximation is at most a factor $n^{-1/3}$ of the side of the square hit, thus the total error is $O(n^{2/3})$. For each group and each segment we define a quotient ρ_i as before, where in the definition we consider the representative point on the representative square of the group. We have a balance associated with every group and every segment, that is a total of $n^{2/3}$ numbers. The balance heuristic works exactly as before. Using similar arguments, one can show then that the distance traversed from the start to the goal is at most $\frac{3}{2}\mathrm{n}+O(n^{2/3})$.

Fejes Toth [5] has shown, albeit by a nonalgorithmic method, that if the straight-line distance from s to t is n, then there is always a path of length $(3n+1)/2$ from s to t that avoids a set of squares of size at most 1; i.e., *existentially* the ratio 1.5 is not only asymptotic but also absolute up to an additive constant. It is an open problem whether this can be achieved algorithmically in an on-line fashion. We leave it also as an open problem whether one can achieve ratio 1.5 in the case of arbitrary squares (unbounded size).

4. The complexity of searching under uncertainty

As it is probably evident from the preceding discussion, finding heuristics that achieve a good ratio to the shortest path in the absence of maps is a formidable problem. Can we use the techniques of Complexity Theory to capture this feeling?

To this end, we must define versions of the problem in which we are given a *partial description* of the map, with several uncertainties remaining (the partial description is necessary, since it will be the input to the corresponding computational problem). As an example of this situation, consider the following problem:

Double-valued Graph: We are given a directed graph $G=(V, E)$, a start node $s \in V$ and a target node $t \in V$. Each edge $(u, v) \in E$ has *two possible lengths* associated with it. Its length is one of the two, and the searcher finds out which one it is only when u is visited. Is there a strategy for traversing the graph, starting from s, ending in t, such that the total distance traversed is no more than r (a given ratio) times the shortest path from s to t in G (with edge lengths consistent with those discovered by the searcher)?

Theorem 4.1. Double-valued Graph *is* PSPACE-*complete.*

Proof. This problem is a two person game for which it is easy to show membership in PSPACE using standard techniques. We will now prove completeness. The reduction is from the quantified satisfiability problem, or QSAT [6]. Consider an instance

$$\exists x_1 \forall x_2 \exists x_3 \ldots F(x_1, x_2, x_3, \ldots, x_n),$$

where F is a Boolean formula in conjunctive normal form with three literals in each clause. Let p be the number of clauses, and let $m=7(n+p)$. The reduction is highlighted in Fig. 9. The graph has less than m nodes, and the arcs have four possible lengths: $1 \ll K \ll L \ll M$; for concreteness we let $K=m^3$, $L=m^6$, $M=m^{12}$. Double values are shown separated by a "$|$". Edges with only one number have only one possible length, and unlabelled edges have length 1. The target ratio r in the instance of the Doubled-value Graph problem is $r=m^4$.

We explain now the construction in detail. First we describe the subgraph defined by the edges that have label 1 or $1|M$. This subgraph consists of two parts: the upper part from the start node s to the node u is a connection in series of gadgets that correspond to the variables $x_1, \ldots, x_n$. We have nodes $x_1, \ldots, x_n$ and x_{n+1}. There is an edge of length 1 from the start node s to node x_1, and a path of two edges from x_{n+1} to u, where the first edge has length 1 and the second has label $1|M$. For every variable x_i there is a corresponding gadget that connects node x_i to node x_{i+1} as shown in the figure. The gadget that corresponds to an existential variable x_i consists of a path of two edges from node x_i to node x_{i+1}, where the first edge has length 1, and the second edge has label $1|M$; see for example the path from node x_1 to node x_2 in Fig. 9. The gadget corresponding to a universal variable consists of two parallel paths of length three where the first and third edge of each path has label $1|M$ and the second edge has length 1; see for example the portion of the graph between nodes x_2 and x_3. The lower part from node u to the target node t is a connection in parallel of paths that correspond to the clauses of the formula F. We have for each clause a path from u to t with seven arcs whose labels are alternatingly 1 and $1|M$. We associate the three arcs of the path labelled $1|M$ with the three literals of the clause.

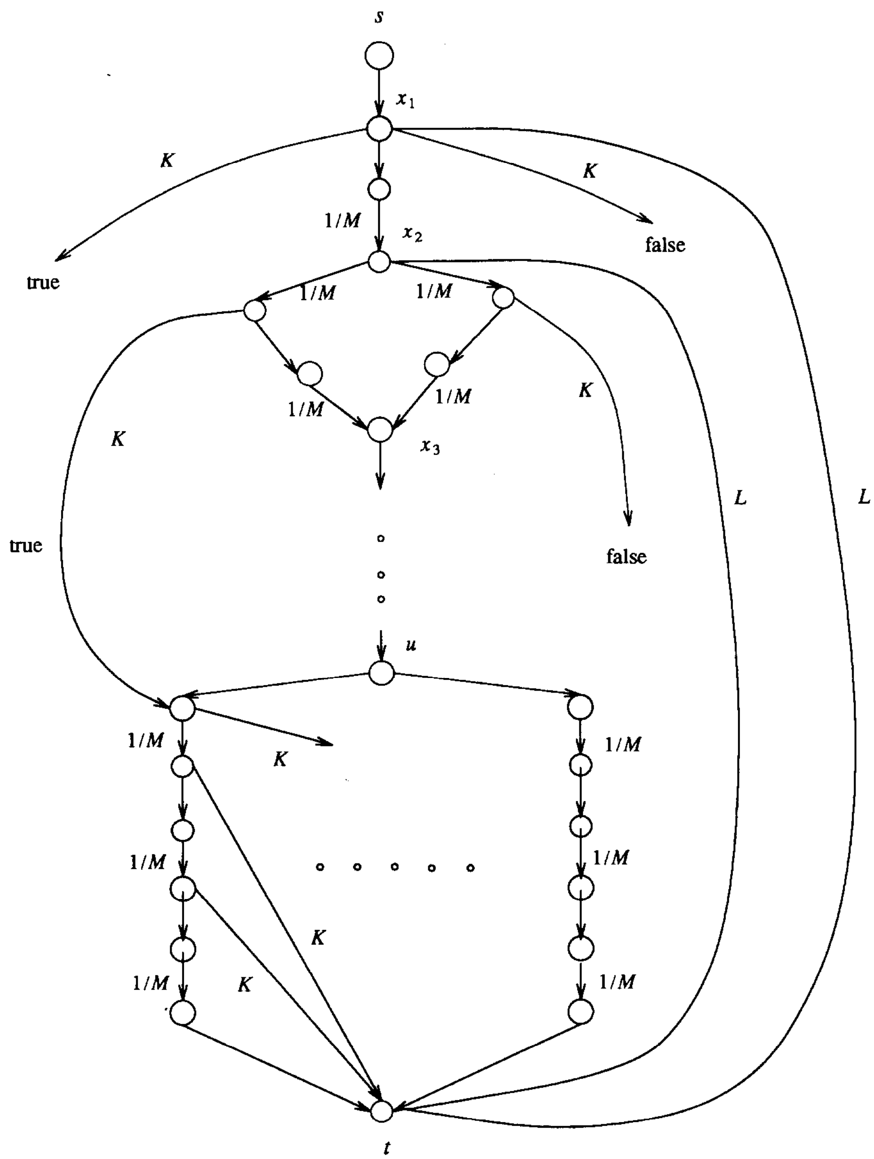


Fig. 9. The reduction.

We describe now the arcs that have length K. Corresponding to each literal x_i or $\overline{x_i}$, the graph contains a path of such arcs. The path corresponding to literal x_i starts from node x_i if x_i is an existential variable, or from the left successor of x_i if x_i is a universal variable, and then goes to the head of the edge in the lower part that corresponds to the first occurrence of the literal x_i in the clauses. From there it goes to the head of the edge that corresponds to the second occurrence of x_i, and so forth, and finally it comes into node x_{i+1}. The path corresponding to a negated literal $\overline{x_i}$ is defined similarly, except that it starts from the right successor of x_i if x_i is a universal variable. In addition to these paths, there are arcs of length K coming into t from the tails of all the $1|M$ edges that correspond to literal occurrences in the lower part. Finally, we include arcs of length L from the nodes $x_1, \ldots, x_{n+1}$ to the target node t. This completes the description of our construction. We claim that the searcher can achieve ratio $r = m^4$ or better against any adversary if and only if the quantified formula Q is valid.

As usual we can view the validity question of the formula Q as a game between an existential player and a universal player who take turns choosing truth values for the variables, the first player chooses for the existential variables and the second player for the universal variables, where the goal of the first player is to make F true, while the goal of the second player is to make F false. If the formula Q is valid then the existential player has a winning strategy, otherwise the universal player has a winning strategy.

Suppose that Q is valid. The searcher imitates the winning strategy of the existential player. From the starting node s, the searcher goes to the first variable node x_1. Suppose that the corresponding variable is existential. Then the searcher chooses the length-K arc out of node x_1 that corresponds to the truth value chosen for the variable x_1 by the winning strategy of the existential player, say for concreteness that $x_1 = \mathit{true}$. When the searcher arrives at the head of the arc with label $1|M$ that corresponds to an occurrence of the literal x_1, the adversary fixes the length of this arc. If the adversary lets this length be 1, then the searcher traverses this arc and then the length-K arc from its tail to the target node t. Clearly, the total distance traversed by the searcher is less than $r = mK$. So, let us assume that the adversary fixes the length of the arc corresponding to the first occurrence of x_1 to be M. Then the searcher goes via a length-K arc to the head of the arc that corresponds to the second occurrence of x_1. Again, if the adversary fixes the length of this arc to be 1, then the searcher can reach the target node as above. The searcher continues visiting the heads of all the arcs that correspond to occurrences of x_1 and the adversary is forced to set the lengths of these arcs to M, making them effectively unusable. Finally the searcher reaches node x_2. If x_2 is an existential variable, the searcher chooses a truth value and proceeds as above.

Suppose that x_2 is a universal variable. The adversary can choose the lengths of the two $1|M$ arcs incident to x_2. If both arcs are set to M, then we observe that s cannot reach t using only length 1 arcs, no matter what lengths the adversary chooses for the remaining arcs. Therefore, in this case the shortest path from s to t has length at least K. The searcher can go from x_2 to t directly via the length-L edge. Clearly, the total distance traversed is less than $mK + L \leq rK$. Thus, for the adversary not to lose, he must choose at least one of the two edges out of x_2 to have length 1, which corresponds to a choice of a truth value for x_2 by the universal player. The searcher traverses this edge, say it goes to the right successor of x_2 corresponding to setting x_2 to false. From there, the searcher goes via the length-K edge to the lower part, and as above follows the path of length-K edges that corresponds to the literal $\overline{x_2}$, while the adversary is forced to choose length M for the edges that are associated with the occurrences of $\overline{x_2}$, if he does not want to lose.

The game between the searcher and the adversary continues in this manner until finally, after the searcher traverses the path corresponding to literal x_n or its negation $\overline{x_n}$, he goes to node x_{n+1}, and from there he traverses the edge to t. The distance traversed until the searcher reaches node x_{n+1} is less than mK, and thus the total distance is certainly less than $mK+L$. Since the truth assignment that has been chosen satisfies the formula F, every path from u to t corresponding to a clause of F contains an edge whose length has been set to M. Therefore, the shortest path from s to t has length at least K.

Conversely, suppose that Q is not valid. First, observe that if the searcher wants to achieve ratio r, then at no point can he traverse an edge of length M because there is always a path from s to t of length $L+1$: from s to x_1 to t. In particular, this means that the searcher cannot visit the head of an arc with label $1|M$ in the upper part that enters a node x_i, because the adversary can then set the length of this arc to M, thereby forcing the searcher to traverse the arc and lose the game. Suppose that the searcher decides at some point to traverse an edge of length L from some node x_i to node t, and that at this point there is still a potential path of length-1 edges from s to t. Then the adversary can set the lengths of all the edges to their lower alternatives, thus obtaining a shortest path of length at most m and preventing the searcher from achieving ratio r.

It follows from the above observations that in order for the searcher to have a chance at winning, he must follow a play of the existential player. When the searcher arrives at a node x_i corresponding to an existential variable, he must choose one of the two length-K edges corresponding to a truth value for x_i. The adversary chooses then M to be the length of the arcs associated with the occurrences of the corresponding literal x_i or $\overline{x_i}$. The searcher is forced to follow the path of length-K edges corresponding to this literal and then go to node x_{i+1}. Suppose that x_{i+1} is a universal variable. Then the adversary fixes the length of the two edges incident to x_{i+1} that have label $1|M$ according to the truth value chosen by the winning strategy of the universal player; if the value is true then the left edge has length 1 and the right edge has length M, otherwise the roles are reversed. By our previous observations, the searcher is forced to follow whichever edge has length 1, and then follow a length-K edge to the first occurrence of the corresponding literal. Since the universal player has a winning strategy, at all times there is a path from u to t in the lower part all of whose edges can still be assigned length 1. Thus, the shortest s–t path has length at most m. At the end of the game the searcher reaches node x_{n+1}, and then he must traverse the edge to t of length L. Consequently, the searcher does not achieve ratio r. Completeness follows. □

It follows easily from the proof that approximating the optimal ratio within any constant is also PSPACE-complete. By a more careful construction we can show:

Corollary 4.2. THE CANADIAN TRAVELLER'S PROBLEM *(as defined in the introduction) is* PSPACE-*complete.*

Proof. The arguments are now a bit more delicate. Before applying the construction of Theorem 4.1, we first form a new set F' of clauses that consists of three copies of the old set, followed by the (tautological) clauses $x_1 \vee \overline{x_1}, \ldots, x_n \vee \overline{x_n}$, followed by another three copies of the old clauses. Let D be the directed graph obtained by applying the construction of Fig. 9 using the new set F' of clauses. Recall that for every literal x_i or $\overline{x_i}$ there is a path of length K edges that traverses the heads of the arcs associated with the occurrences of the literal. Modify the graph so that all literal paths have the same length $T = 6mK$, by introducing an equal number of length-K edges at the beginning and the end of each literal path. We also change the label of the edge (x_{n+1}, t) into $1|L$. Let D' be the resulting directed graph.

We will first obtain from D' an undirected graph G with double-valued edges. Consider the graph of Fig. 10 where there are M parallel paths from a to b. Clearly, the adversary can play so that the searcher cannot tell with certainty whether there is an a–b path of length 3 unless he tries all M paths and thus travels distance M. We substitute in D' a copy of this gadget in place of each path that consisted of a length-1 edge followed by a $1|M$ edge to a node x_{i+1} or node u. Also, we replace every length-1 edge from the lower part by a copy of this gadget. The effect of these replacements will be that the searcher cannot use these edges. We make now all remaining edges of D' undirected and let G be the resulting graph. Let d be the shortest s–t path if the adversary sets all edges to their lower alternatives. Note that the shortest path uses only length-1 edges from s to x_{n+1} and then to t, and thus d is between $3n$ and $4n$. Let the target ratio r be $n(T+2)/d$.

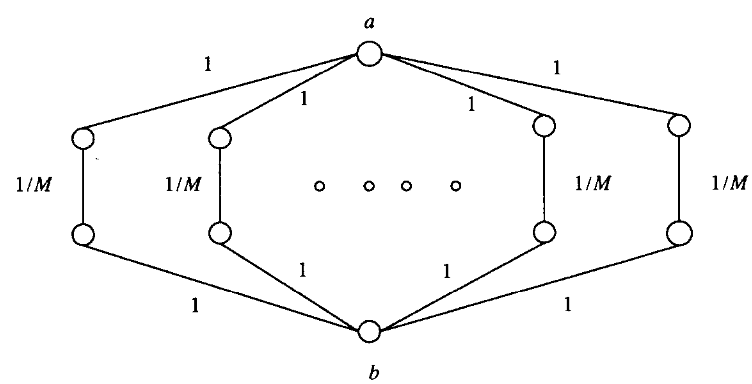


Fig. 10. Elimination of directed edges.

Suppose that the quantified formula Q is valid. As before, the searcher simulates the winning strategy of the existential player. He traverses distance $T+1$ for each variable, as long as the adversary plays "normally". If the adversary chooses length

1 for an arc of the lower part corresponding to an occurrence of a true literal, or for the arc (x_{n+1}, t), then the searcher reaches t with distance less than $n(T+2)$ and thus achieves the ratio r. If the adversary chooses for some universal variable x_i to set both of its incident edges to M, then the shortest s–t path has length at least K. The searcher can use the edge from x_i to t and the total distance traversed is no more than $n(T+1)+L \leq rK$. Finally, if the searcher arrives at x_{n+1}, either the adversary lets the edge to t have length 1, in which case the distance traversed is at most $n(T+2)$ and the shortest path has length d, or the adversary chooses length L in which case the shortest path has length K because the searcher followed the winning strategy of the existential player.

Suppose that the formula Q is not valid. The arguments are similar to Theorem 4.1. The difference from the directed case is that now the searcher can backtrack. One danger is that he may follow the path of a literal x_i only partway, and then backtrack. The trivial clause $x_i \vee \overline{x_i}$ in the middle forces him to travel at least distance T in order to break the u–t path corresponding to this clause, i.e., force the adversary to set an edge of the path to length M. Another danger is that the searcher may follow (at least parts of) both the paths corresponding to x_i and its negation $\overline{x_i}$ to force the adversary to break (set to M) the corresponding arcs from the lower part, in order to effectively satisfy the clauses of F. The point is that unless he spends distance $4T/3$, the innermost copies of the original clauses remain intact for at least one of the truth values.

Consider now what happens when the searcher arrives at node x_{n+1}. If he has travelled distance T for every variable and has travelled at least $4T/3$ for some variable, then he has used distance $(n+\frac{1}{3})T$ which exceeds rd. The adversary chooses length 1 for the edge to t, thus ensuring an s–t path of length d. On the other hand, if the searcher has travelled less than $4T/3$ for any variable or has not travelled T for some variable, then there is still a path of length-1 edges in the lower part from u to t. The adversary lets the length of the edge (x_{n+1}, t) be L, thus forcing the searcher to use distance at least L, while the shortest path has length less than m. In either case, the searcher cannot achieve ratio r.

Finally, we can construct from the graph G an instance G' of the CANADIAN TRAVELLER'S PROBLEM where every edge has length 1 and some edges have a question mark. Note that all the edge lengths in G are polynomially bounded. Every single-valued edge of G is replaced in G' by a path of the appropriate length. All double-valued edges of G are $1|M$ or $1|L$. If (a, b) is a double-valued edge of G, then we replace it by an edge (a, b) of length 1 with a question mark and a parallel path from a to b of length M or L (and no question mark). □

Consider next the following geometric problem:

3-D OBSTACLE SCENE: We are given a three-dimensional scene, with obstacles that are (possibly intersecting) rectilinear blocks. However, certain special obstacles *may not be present.* We find out whether they are only by visual contact. Again, we wish to find a search strategy that achieves a given ratio.

Corollary 4.3. 3-D OBSTACLE SCENE *is* PSPACE-*complete.*

Proof. The reduction is from DOUBLE-VALUED GRAPH. We embed the graph in 3-space[1]. We implement edges as parallel "tunnels" with walls formed with obstacles joining two nodes. Edge lengths can be implemented by having tunnels that zig-zag as required. Double-valued edges are implemented as parallel tunnels of the appropriate lengths, the shortest of which is obstructed by a (possibly absent) obstacle. This obstacle is visible from both endpoints of the edge (it is easy to see that a graph can be embedded in 3-space so that this condition is observed). □

Finally, suppose that the length of each edge of a graph has a given discrete probability distribution. When we arrive at a node we discover the actual length of its incident edges. We wish to devise a strategy that minimizes the *expected* ratio to the optimum path. Or we may want a strategy that minimizes the expected distance traversed from the start to the goal node. For these interesting problems we can show the following:

Theorem 4.4. *The stochastic optimization problems mentioned above can be solved in polynomial space, and are* #P-*hard.*

Proof. Membership in PSPACE follows from the fact that this is a "game against Nature" [14]. We show #P-hardness for the problem of computing the expected cost of an optimal strategy, where cost is the total distance or the ratio to the shortest path. With simple modifications one can easily prove that it is also #P-hard to compute the optimal strategy itself (for example, to compute the best move out of the start vertex). Our reductions are from the s–t reliability problem: Given an undirected graph G with two distinguished nodes s and t, the s–t reliability is the probability that s is connected to t assuming that the edges of the graph fail (i.e., are deleted) independently with probability $\frac{1}{2}$ [6]. We will show #P-hardness first for the minimization of the expected distance traversed, and then will show it for the ratio criterion.

Let n be the number of nodes and e the number of edges of G. Let H be the graph obtained from G by letting every edge of G have, with probability $\frac{1}{2}$, length 1 or infinity (i.e., an appropriately large number), and by adding an edge from s to t which has length $M = 4n2^e$ with probability 1. Let g be the s–t reliability of G, and let l be the expected distance traversed by the optimal search strategy. As we will show, if we know l, then we can derive g.

[1] If we could make the graph in the reduction of Theorem 4.1 planar, which at present we cannot, then the 2-dimensional problem could also be proved PSPACE-complete.

Observe that g is equal to $k/2^e$ for some integer k. From the definitions, we have $l \geq M(1-g)$, because if s is not connected to t, using edges of length 1, then we must traverse the new edge (s, t). On the other hand the following strategy implies that $l \leq M(1-g)+g2n$. First explore starting from s the graph G using only length-1 edges in a depth-first manner, and backtracking when necessary. It is easy to see that after traversing distance no more than $2n$, we can either find a path to t through the graph G, or backtrack to s having determined that no such path exists. Combining the two inequalities, we conclude that $2^e - k \leq l/4n \leq 2^e - k + \frac{1}{2}$, that is, $k = 2^e - [l/4n]$.

For the ratio criterion, form a graph H' as above, except that instead of adding the edge (s, t) we add a new start vertex s', an edge from s to s' of length $H = 4n2^{2e}$ and an edge from s' to t of length $L = 8n2^{3e}$. One strategy for the searcher is to go directly from s' to t. If the outcome of the random experiment is such that s is disconnected from t in G, then the shortest s'-t path has length L and the ratio is 1, otherwise the shortest path has length between $H+1$ and $H+n$ and the ratio is approximately $\frac{L}{H}$. Thus, the expected ratio of this strategy is approximately $(1-g)+g(\frac{L}{H})$. A second strategy for the searcher is to go to s and try to find a path through G, and if he does not succeed, to backtrack to s' and follow the length L edge to t. This strategy has expected ratio approximately $g+(1-g)(2H+L)/L$ and thus is better since $H \ll L$. The optimal expected ratio r satisfies the following inequalities: $r \geq 1+(1-g)(2H/L)$ and $r \leq 1+g2n/H+(1-g)(2H+2n)/L$. Let $g = k/2^e$. Then it is easy to see that the integer part of $r2^{2e}$ is equal to $2^{2e}+2^e-k$. □

References

[1] R.A. Baeza-Yates, J.C. Culberson and G.J.E. Rawlins, Searching with uncertainty, in: *Proc. Scandinavian Workshop on Algorithm Theory* (1988) 176–189.

[2] A. Borodin, N. Linial and M.E. Saks, An optimal online algorithm for metrical task systems, *SIAM J. Comput.*, to appear.

[3] J. Canny and J. Reif, New lower bound techniques for robot motion planning problems, *Proc. FOCS* (1987) 49–60.

[4] S.E. Dreyfus, An appraisal of some shortest path algorithms, *Operations Research* **17** (1969) 395–411.

[5] G. Fejes Toth, Evading convex discs, *Studia Sci. Math. Hungar.* **13** (1978) 453–461.

[6] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness* (Freeman, New York, 1979).

[7] E.L. Lawler, *Combinatorial Optimization: Networks and Matroids* (Holt, Rinehart and Winston, New York, 1977).

[8] M. Manasse, L. McGeoch and D. Sleator, Competitive algorithms for on-line problems, in: *Proc. 20th STOC* (1988) 322–333.

[9] J.S.B. Mitchell, Shortest paths in the plane among obstacles, Technical Report, Stanford OR, 1986.

[10] J.S.B. Mitchell, Planning shortest paths, PhD dissertation, Stanford OR, 1987.

[11] J.S.B. Mitchell, D. Mount and C.H. Papadimitriou, The discrete geodesic problem, *SIAM J. Comput.*, 1987.

[12] J.S.B. Mitchell and C.H. Papadimitriou, The weighted regions problem, to appear in *J.ACM* (1991).

[13] C.H. Papadimitriou, An algorithm for shortest-path motion in three dimensions, *Inform. Process. Lett.* **20** (1985) 259–263.

[14] C.H. Papadimitriou, Games against nature, J. Computer System Sci. **31**(2) (1985) 288–301.

[15] C.H. Papadimitriou, Shortest path motion, in: *Proc. 1987 FST-TCS Conf. New Delhi* (1987).
[16] C.H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity* (Prentice-Hall, Englewood Cliffs, NJ, 1982).
[17] J. Pearl, *Heuristics* (Addison-Wesley, Reading, MA, 1984).
[18] D. Sleator and R.E. Tarjan, Amortized efficiency of list update and paging rules, *Comm. ACM* **23** (1985) 202–208.