Binary Space Partitions of Orthogonal Subdivisions*

John Hershberger† Subhash Suri‡ Csaba D. Tóth§

Abstract

We consider the problem of constructing binary space partitions (BSPs) for orthogonal subdivisions (space filling packings of boxes) in d-space. We show that a subdivision with n boxes can be refined into a BSP of size $O(n^{\frac{d+1}{3}})$, for all $d \geq 3$, and that such a partition can be computed in time $O(K \log n)$, where K is the size of the BSP produced. Our upper bound on the BSP size is tight for 3-dimensional subdivisions; in higher dimensions, this is the first nontrivial result for general full-dimensional boxes. We also present a lower bound construction for a subdivision of n boxes in d-space for which every axis-aligned BSP has $\Omega(n^{\beta(d)})$ size, where $\beta(d)$ converges to $(1+\sqrt{5})/2$ as $d \to \infty$.

1 Introduction

Many algorithms in computational geometry, robotics, and computer graphics decompose a set of objects and the ambient space for efficient processing of certain queries. A binary space partition (BSP) is a popular scheme for constructing such decompositions. Given an open convex region of space containing a set of pairwise disjoint objects S, a BSP partitions the region and objects with a cutting hyperplane, then recursively partitions the two resulting subproblems. The process stops when each open partition region intersects at most one object of S. See Figure 1 for an example in two dimensions. In the ideal case, the number of regions in the final BSP would be at most n, the number of input objects. In general, however, the recursive partitioning may fracture input objects many times, and the size of the partition can be much larger than n.

Binary space partitions were introduced in the computer graphics community [10, 16] to solve hidden surface removal problems. But they are now used for a wide variety of applications, including set operations in solid modeling, visibility preprocessing for interactive walkthroughs, shadow generation, and cell decomposition methods in robotics, to name only a representative sample [2, 6, 7, 12, 13, 17]. The *size* of a BSP is the total number of pieces the input objects are partitioned into by the BSP, and it is a measure of the *fragmentation* caused by the partition. Because BSPs are often used to decompose large data sets, their size can be crucial to the performance of the applications that rely on them. Theoretical analyses have therefore focused primarily on BSP algorithms that produce small size partitions.

*A preliminary version of this work appeared in the Proceedings of the 20th Annual ACM Symposium on Computational Geometry (Brooklyn, NY, 2004), ACM Press, 2004, pp. 230-238.

†Mentor Graphics Corp., 8005 SW Boeckman Road, Wilsonville, OR 97070, and (by courtesy) Department of Computer Science, University of California at Santa Barbara. Email: john_hershberger@mentor.com.

‡Department of Computer Science, University of California at Santa Barbara, CA 93106, suri@cs.ucsb.edu. This author's research on this paper was partially supported by National Science Foundation grant CCR-9901958.

§Department of Mathematics, MIT, Cambridge, MA 02139, toth@math.mit.edu. The research on this paper was performed while this author was a post-doctoral fellow at the University of California at Santa Barbara.

A theoretical study of BSPs began in earnest with two influential papers by Paterson and Yao [14, 15]. In two dimensions, Paterson and Yao gave an $O(n)$ size BSP construction for orthogonal (axis-parallel rectangular) objects, and an $O(n \log n)$ size construction for general polygons with a total of n edges. Recently, Tóth [18] has shown an almost-matching lower bound of $\Omega(n \log n / \log \log n)$ for arbitrary polygons in the plane. In three dimensions, Paterson and Yao gave an $O(n^{3/2})$ size BSP construction for orthogonal boxes, and an $O(n^2)$ size construction for arbitrarily oriented polyhedra with a total of n edges. They also gave lower bound constructions matching these upper bounds.

The results of Paterson and Yao have been extended and improved in several important directions, yet the complexity of BSPs in three and higher dimensions is still not fully understood. We briefly review the previous research most relevant to our work. De Berg shows that any *uncluttered* scene of n objects in d-space admits a linear size BSP [4]; informally, "uncluttered" means that any cubical region intersecting more than a constant number of objects must include a vertex of the bounding box of one of the objects. For instance, a collection of hypercubes (or, more generally, of *fat* objects) is uncluttered. Such an assumption may be practical in some situations, but it seems quite restrictive in general.

Paterson and Yao [14] consider the complexity of BSPs for 1-dimensional objects (line segments) in d-space, and show that a worst-case set of n axis-aligned line segments requires a BSP of size $\Theta(n^{d/(d-1)})$, for $d > 2$. Dumitrescu, Mitchell, and Sharir [8] give an upper bound of $O(n^{d/(d-k)})$ for the BSP size of disjoint k-dimensional orthogonal objects in d-space for $1 \leq k < d$. This bound is asymptotically tight for $k < d/2$. The only known tight bound for orthogonal k-flats in d-space with $k \geq d/2$ is the case $d = 4$ and $k = 2$ where a $\Theta(n^{5/3})$ bound has been shown [8].

The general $O(n^{d/(d-k)})$ upper bound does not say anything about full-dimensional boxes. Berman, DasGupta, and Muthukrishnan [5] show that every set of n axis-aligned rectangles in the plane has a BSP of size at most $3n$. The best known lower bound, $\frac{7}{3}n - o(n)$, is due to Dumitrescu, Mitchell, and Sharir [8]. If the rectangles tile the plane, however, then Berman, DasGupta, and Muthukrishnan [5] prove an upper bound of $2n - 1$, matching a lower bound of $2n - o(n)$ by Dumitrescu, Mitchell, and Sharir [8] (originally designed for axis-parallel line segments). For full-dimensional axis-aligned boxes in d-space, $d \geq 2$, one can obtain an upper bound of $O(n^{d/2})$ by using the result of [8] in conjunction with an observation of Paterson and Yao [15] that a set of n full-dimensional boxes in d-space has the same BSP complexity in certain cases as the set of their $(d-2)$-dimensional faces.

Agarwal et al. [1] consider BSPs of 2-dimensional *fat rectangles* in 3-space—each rectangle has sides parallel to the axes and a constant aspect ratio. They show that a collection of n such rectangles admits a BSP of size $n2^{O(\sqrt{\log n})}$. This bound was recently improved to $O(n \log^8 n)$ by Tóth [19].

Our Results

In an attempt to understand the true complexity of d-dimensional BSPs, we consider a natural but restricted setting: *orthogonal subdivisions.* An orthogonal subdivision is a collection of interior-disjoint rectilinear boxes that fill their containing space. (That is, a subdivision is a space-filling packing of boxes.) Our interest in subdivisions is motivated by the observation that all the known lower bound constructions involve intertwined rod-like objects that create many "holes." This raises a natural question: what is the complexity of the BSP for scenes where the complement space can be "tiled" with few boxes, say a linear number? Our main result is the following theorem:

> *Given an orthogonal subdivision with n boxes in d-space, we can construct a BSP for it of size $O(n^{\frac{d+1}{3}})$, for any $d \geq 2$.*

Thus, for collections of orthogonal objects that fill their containing space, we are able to improve the worst-case bound on the BSP size from $O(n^{d/2})$ to $O(n^{\frac{d+1}{3}})$. As a corollary, for any collection of

n orthogonal boxes whose complement space can be tiled with m boxes, there exists a BSP of size $O((n+m)^{\frac{d+1}{3}})$. Our subdivision BSP can be constructed in time $O(K \log n)$, where K is the output size. We also exhibit a lower bound construction for an important class of BSPs: An *axis-aligned BSP* is a BSP in which every cutting hyperplane is orthogonal to one of the axes. We describe a subdivision that requires an axis-aligned BSP of size $\Omega(n^{\beta(d)})$ in $\mathbb{R}^d$, where $\beta(d)$ converges to $(1+\sqrt{5})/2$ as d goes to infinity. The value of $\beta(d)$ is $4/3$ for $d=3$, and thus our upper bound is tight in 3-space.

Our BSP algorithms are quite simple and therefore easy to implement. Their key component is a round robin partitioning scheme (in which cutting hyperplanes orthogonal to the coordinate axes are selected in a round robin order). They follow a conventional scheme—a round robin partitioning phase followed by efficient BSP construction in the terminal regions. Such a two-phase partitioning framework has been used earlier in several papers [1, 8], but in each case it requires problem-specific insights to find the right stopping rule for the round robin phase as well as an analysis for the terminal case. In our case, we show that "round robin cutting until no interior $(d-3)$-dimensional face remains" is a good stopping rule. One tricky part of analyzing this round robin scheme is that each cut may also *increase* the number of $(d-3)$-faces. The second phase requires efficient binary space partitions for regions that do not contain a $(d-3)$-face. To this end, we prove the following theorem, which may have independent appeal:

> *Consider a box R in d-space, and a subdivision of R into n boxes such that no $(d-3)$-dimensional face of any box intersects the interior of R. Then there is a BSP of size $O(n)$ for this subdivision.*

2 Geometric Preliminaries

A *d-dimensional box* B is the cross product of d real-valued intervals. Given a box R and a set of boxes $S=\{B_1,\ldots,B_n\}$, we say that S is a *subdivision* of R if each B_i lies in R, the union of the B_i's covers R, and the B_i's have pairwise disjoint interiors. Thus, an orthogonal subdivision of R is a packing by boxes that completely fills R. In this paper, we consider BSPs for d-dimensional box subdivisions only.

A BSP for problem instance (R,S) partitions R with a hyperplane into sub-boxes R_1 and R_2, and recursively solves the problems (R_1,S_1) and (R_2,S_2), where $S_i = S \cap R_i$ is the set of fragments of the input objects contained in R_i, for $i=1,2$. The partitioning stops when every subproblem contains at most one object (fragment). A BSP is naturally modeled as a binary tree: the root corresponds to the problem (R,S), and its two children correspond to the subproblems (R_1,S_1) and (R_2,S_2). Every internal node stores the splitting hyperplane for the corresponding problem; the leaf nodes correspond to the regions in the final partition. See Figure 1 for a small example in two dimensions. For ease of reference, we will often call R the *container box* for the problem instance. Thus, R_i is the container box for the ith subproblem, for $i=1,2$, produced at the root.

Free cuts are important for the construction of small BSPs. A free cut is a hyperplane that separates the object set without splitting any object. When a subproblem (R,S) has a free cut, it is always worth splitting the subproblem with the free cut, since the split does necessary work—separating objects that must be separated by the BSP—without increasing the complexity of the subproblems (and hence the final BSP size). In Figure 1, the cuts along h_2, h_3, h_4, and h_5 are free cuts.

A d-dimensional box B has 2^d vertices, $d2^{d-1}$ edges, and $2d$ facets. B also has many faces of intermediate dimensions j, for $2 \le j \le d-2$. A *k-face* of B is a k-dimensional box, and it can

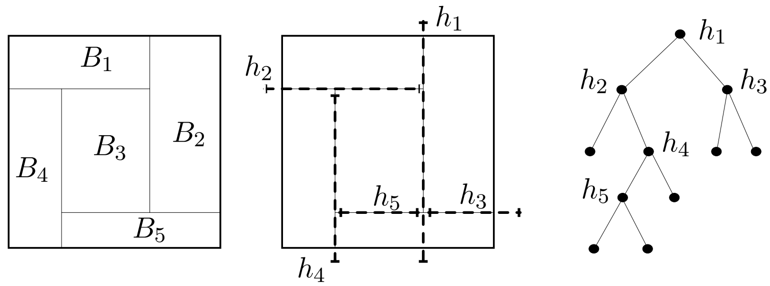

Figure 1: An orthogonal subdivision S (left), a BSP for S (middle), and the corresponding binary tree (right).

be characterized as follows: a k-face of B is obtained from the cross-product B by fixing $(d-k)$ coordinates at one of the two extremes of the corresponding intervals; the remaining k coordinates maintain the same extent as B. Thus, vertices are the 0-faces, edges are the 1-faces, and facets are the $(d-1)$-faces of B. It is easy to see that every k-face of the box B, for $0 \le k < d$, lies on the boundary of B. Our algorithm exploits the structure of boxes whose extents in certain dimensions match those of the container box. We therefore introduce the notions of *pass-through* and *k-rod*.[1]

Consider a container box R, and a box B in a subdivision of R. We say that B is *pass-through* for R in dimension j if the extent of B in dimension j equals that of R. We say that box B is a *k-rod* in R if B is pass-through in exactly k dimensions. If B is a k-rod in R, then its *orientation* $\sigma(B)$ is the set of the k dimensions in which B is pass-through for R. Whenever the container box R is clear from the context, we will simply say that B is a k-rod, without mentioning R. Figure 2 illustrates k-rods in three dimensions, for $k = 1, 2$.

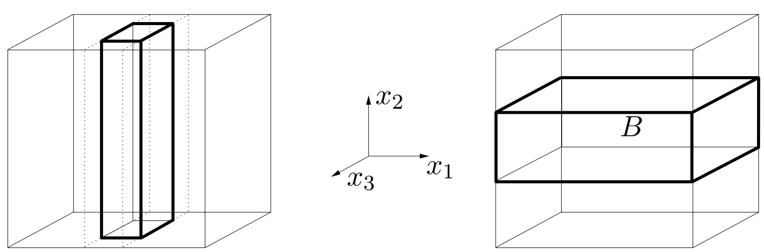

Figure 2: A 1-rod of orientation $\{x_2\}$ on the left and a 2-rod of orientation $\{x_1, x_3\}$ on the right in $\mathbb{R}^3$.

Suppose B is a rod in a box subdivision of some container R. If B is a d-rod, then, of course, B fills up R, and it is the only box in R. If B is a $(d-1)$-rod, with orientation $\sigma(B) = \{x_1, x_2, \ldots, x_{d-1}\}$, then at least one of the two facets of B orthogonal to the x_d-axis intersects the interior of R, separating it into two parts. A cut along this facet is a free cut for the subdivision of R. For instance, in the right half of Figure 2, the box B is a 2-rod in 3-space, and there are free cuts along its top and bottom faces. We summarize this fact for future reference.

Lemma 2.1. *If a box subdivision contains a $(d-1)$-rod, then there is a free cut along one of the facets of that rod.*

[1]The notion of pass-through also appears in [8]. Our analysis, however, requires a more detailed investigation into the properties of rods.

We next establish an elementary but useful characterization of k-rods: a box B is a k-rod in a container R *if and only if* every j-face of B is disjoint from the interior of R, for all $j < k$. For instance, in Figure 2 (right), B is a 2-rod, and all of its vertices and edges (0- and 1-faces) lie on the boundary of the container box. We call a face an *interior face* if it intersects the interior of the container box.

Lemma 2.2. *Suppose B is a box in the subdivision of R. Then B is a k-rod if and only if B has no interior j-faces for $j < k$ and at least one interior j-face for every j such that $k \le j \le d$.*

Proof. Consider a k-rod B. Let F be one of its j-faces, where $j < k$. By definition, F has a fixed coordinate in $(d - j)$ dimensions, and the same extent as B in the remaining j dimensions. Because B is a k-rod, it is pass-through for R in k dimensions, and $k \ge j + 1$. Thus, at least one of the fixed coordinates of F is at one of the extremes of a pass-through dimension of B, and so F lies on the boundary of R. Now consider j such that $k \le j \le d$. Choose F to be a j-face whose varying coordinates include all the pass-through directions of the k-rod B, and whose fixed coordinates are chosen inside the corresponding intervals of the cross product of the container box R (possible because B is pass-through in none of those $d - j \le d - k$ dimensions). By construction, this j-face F intersects the interior of R. □

3 An Upper Bound in $\mathbb{R}^3$

Our first result is an optimal binary space partition for 3-dimensional subdivisions. Our BSP has worst-case size $O(n^{4/3})$, which is optimal because a modified Paterson-Yao construction gives a matching lower bound for orthogonal subdivisions (see Section 6). The 3-dimensional bound is central to our main result, because our algorithm for constructing the d-dimensional BSP uses projection of the d-dimensional subdivision onto an appropriate 3-space (after a suitable number of round-robin cuts).

Our algorithm (Algorithm 3-BSP) is a round robin partitioning scheme that iteratively slices the subdivision by planes passing through interior box vertices, producing a collection of smaller subdivisions, each of which ultimately contains only j-rods, for $j \ge 1$. A similar round robin scheme is also used by Murali [11] in his construction of BSPs for fat axis-aligned rectangles in $\mathbb{R}^3$. Interestingly, while Murali also achieves an $O(n^{4/3})$ size BSP, that complexity is suboptimal for his problem. In our case, the round robin algorithm produces an optimal BSP.

The partition tree T created by Algorithm 3-BSP is *not* a BSP—the cells output by the algorithm may contain multiple boxes. These terminal regions, however, do not contain any vertices of the subdivision. We can refine each such box into a proper BSP with only a constant factor increase in complexity, and append the tree associated with this refinement to each leaf of T. Therefore, the key to efficiency is bounding the total complexity of all the regions produced by the slicing procedure.

Suppose we have a container box R and its subdivision $S = \{B_1, B_2, \ldots, B_n\}$ into n boxes, so that there are m vertices in the interior of R; clearly, $m \le 8n$. Our partitioning scheme cuts the subdivision using planes normal to each of the three axes in turn, cycling through the three possible orientations in a round robin fashion. In the following pseudo-code description of our algorithm, we use the term *median x_j plane* of a point set to denote the plane normal to the x_j-axis and passing through the median x_j coordinate of the point set.

Algorithm 3-BSP

- Input is the box subdivision $S = \{B_1, B_2, \ldots, B_n\}$ of a container box R.

- Initialize $i = 0$, and $\mathcal{C}_0 = \{R\}$.
- While there is a container box $C \in \mathcal{C}_i$ with a subdivision vertex in its interior, do
  1. For each container box $C \in \mathcal{C}_i$ with at least one vertex of the subdivision in its interior, split C by the median x_p plane of the vertices in the interior of C, where $p = 1 + (i \bmod 3)$.
  2. Apply all possible free cuts.
  3. Let $\mathcal{C}_{i+1}$ be the set of all container boxes resulting from these cuts, and set $i := i + 1$.
- Return $\mathcal{C}_i$.

Using an argument similar to one of Paterson and Yao [15] and Murali [11], we derive an upper bound on the total number of box fragments generated by our algorithm by considering how many times the edges of the original subdivision S are cut.

Lemma 3.1. *Consider the partition tree T generated by Algorithm 3-BSP. At depth i in T, there are $O(n2^{i/3})$ fragments of the original edges of S.*

Proof. Each edge is parallel to one of the three directions x_1, x_2, and x_3. It can be cut only by planes orthogonal to its direction. Since our splitting planes cycle through the three directions, an edge can be cut at every third level of T. Thus, the number of fragments of a given original edge at most doubles at every third level. Since the number of edges in the subdivision at the root of T is $O(n)$, the total number of edge fragments at depth i is $O(n2^{i/3})$. □

Lemma 3.2. *Suppose S is a 3-dimensional box subdivision with n boxes, and $m = O(n)$ vertices lie in the interior of the container box. Then the partition created by Algorithm 3-BSP produces $O(nm^{1/3}) = O(n^{4/3})$ fragments of the input boxes.*

Proof. We analyze the algorithm in *rounds* of three consecutive steps: round j corresponds to the steps $i = 3j, 3j+1, 3j+2$. In one round, cuts are made in all three directions, and a box of a subproblem can be split into at most eight pieces.

We observe that if a box is split in a subproblem, then at least one edge of the box must be interior to the container box for this subproblem. This is because only a 3-rod or a 2-rod can have all its edges on the boundary of the container box (Lemma 2.2), but no 3-rods are ever split and all 2-rods are eliminated in step 2 by free cuts. Lemma 3.1 implies that $O(n2^j)$ box fragments can be further partitioned at round j. Since there is no vertex in the interior of any container cell after $\lceil \log m \rceil$ steps, the algorithm terminates in $\lceil \frac{1}{3} \log m \rceil$ rounds. The number of box fragments produced is

$$O\left(n \cdot \sum_{j \leq \lceil \frac{1}{3} \log m \rceil} 2^j\right) = O\left(n \cdot 2^{(\log m)/3}\right) = O\left(nm^{1/3}\right) = O\left(n^{4/3}\right).$$

□

All that remains now is to show that the terminal cells output by Algorithm 3-BSP can be refined into linear size BSPs. The interior of each of these terminal cells is empty of subdivision vertices, and by Lemma 2.2, every box restricted to these containers is a k-rod, $k \geq 1$. We first establish a useful technical lemma for the case where all subdivision boxes are 1-rods.

Lemma 3.3. *If all boxes in a 3-dimensional subdivision are 1-rods, then taken together the 1-rods have at most two distinct orientations.*

Proof. Suppose to the contrary that A, B, and C are three 1-rods with three distinct orientations. Let ℓ_A, ℓ_B, and ℓ_C be lines running down the centers of A, B, and C, respectively, parallel to their rods' orientations. These are three axis-parallel skew lines. Let P_{AB} be the axis-aligned plane containing ℓ_A and intersecting ℓ_B (see Figure 3). Define P_{BC} and P_{CA} similarly. Note that P_{AB} does not intersect ℓ_C because it is parallel to it. The point $P_{AB} \cap \ell_B$ lies in the interior of B—P_{AB} lies between the planes supporting the two opposite faces of the container box that ℓ_B pierces. Now $P_{AB} \cap P_{BC}$ is a line parallel to ℓ_A that intersects ℓ_B. It follows that $P_{AB} \cap P_{BC}$ is disjoint from A and intersects B. Similar claims hold for $P_{BC} \cap P_{CA}$ and $P_{CA} \cap P_{AB}$.

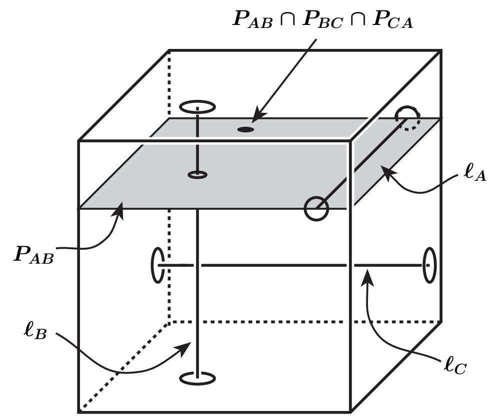

Figure 3: 1-rods with three different orientations imply the existence of an interior vertex.

Let p be the point $P_{AB} \cap P_{BC} \cap P_{CA}$. Point p is disjoint from $A \cup B \cup C$, and p also lies in the interior of the container box, since it is connected by three axis-parallel lines to the points $P_{AB} \cap \ell_B$, $P_{BC} \cap \ell_C$, and $P_{CA} \cap \ell_A$, all of which lie in the interior of the container box. Now we claim that the box containing p cannot be a 1-rod in this subdivision: for each of the axis-parallel directions there is a line (e.g., $P_{AB} \cap P_{BC}$) that intersects one of A, B, and C, and hence the box containing p cannot touch any pair of opposite faces of the container box. This completes the proof. ☐

Lemma 3.4. *Consider a set of boxes $S = \{B_1, B_2, \ldots, B_n\}$ that forms a subdivision of a 3-dimensional container box R. If none of the vertices of any box B_i is in the interior of R, then there is a BSP of size at most $2n - 1$ for S.*

Proof. We proceed by induction on n. The base case, $n = 1$, is trivial. Now suppose that Lemma 3.4 holds for every n', $1 \leq n' < n$. If no 0-face of any box B lies in the interior of R, then every box is a rod by Lemma 2.2. A 3-rod completely fills the container, thus $n = 1$ and no BSP cuts are needed. If there is a 2-rod, then there is a free cut by Lemma 2.1: We split R into two subdivisions along the free cut such that each has fewer than n boxes, and induction completes the proof. We may assume, therefore, that all boxes are 1-rods. By Lemma 3.3, all rods in the subdivision have at most two different orientations. We handle the two cases below separately.

If the rods in the subdivision have two different orientations, we show that there is a free cut, and the proof then follows by induction, since each subproblem produced by the free cut is strictly smaller than n. Suppose, without loss of generality, that the orientation of every rod is either $\{x_1\}$ or $\{x_2\}$. Observe that all the 1-rods properly intersecting a hyperplane H orthogonal to x_3 must have the same orientation, or else two rods with different orientations would intersect on H. (This observation is equivalent to applying Lemma 4.1 to the 2-dimensional subdivision on H.) Now consider a point p

on the common boundary of two 1-rods of different orientations, and let H be a hyperplane passing through p and orthogonal to x_3. Any (closed) box of S intersecting H lies on one side of H because there are 1-rods of different orientations on the two sides of H. Every box intersecting H has a face along H, and therefore H is a free cut.

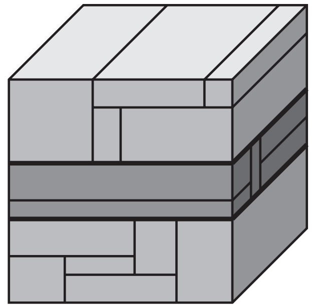

Figure 4: 1-rods with two different orientations imply the existence of a free cut.

If all the 1-rods in the container have the same orientation, then we project them onto the plane orthogonal to their orientation. This gives a two-dimensional subdivision, which has a two-dimensional BSP of size at most $2n-1$ by a result of Berman, DasGupta, and Muthukrishnan [5]. Extending each linear cut of the two-dimensional BSP into a planar cut parallel to the rod orientation, we obtain a BSP for S_A of size at most $2n-1$. □

Thus, combining Lemmas 3.2 and 3.4, we get the main result of this section.

Theorem 3.5. *A 3-dimensional box subdivision with n boxes and m interior vertices has a BSP of size $O(nm^{1/3}) = O(n^{4/3})$. We can construct such a BSP in time $O(\min(nm^{1/3}, K\log n))$, where K is the size of the final BSP.*

The implementation that achieves the $O(\min(nm^{1/3}, K\log n))$ construction time is quite simple. We maintain all the extents of all the boxes in all containers. It is easy to determine from the extents which boxes are split into two, where the median planes lie, and which faces are free cuts. If we sort the extents initially in all dimensions, we can find medians trivially in linear time, and can create sorted extent lists for the child containers in the same time. We spend $O(n\log n)$ time for the initial sorting and $O(K)$ time for each of the $O(\log n)$ steps. Because $K \geq n$, this gives a running time of $O(K\log n)$.

The same implementation also has a running time of $O(nm^{1/3})$—without a logarithmic factor. By Lemma 3.1, the number of box fragments at step i is $O(n2^{i/3})$. The time for step i is therefore $O(n2^{i/3})$ as well. Because the time per step increases geometrically, the last step dominates the running time, giving a bound of $O(nm^{1/3})$.

The size upper bound of $O(n^{4/3})$ in Theorem 3.5 is tight in the worst case: In Section 6, we exhibit a 3-dimensional subdivision requiring a BSP of size $\Omega(n^{4/3})$.

4 The Structure of Rods in $\mathbb{R}^d$

In order to extend our 3-dimensional BSP algorithm to higher dimensions, we need two key ingredients: (1) a suitable stopping rule for our round robin algorithm, that is, one that does not result in too much fragmentation, and (2) a linear-size BSP construction for the cells obtained when the round robin phase ends. In three dimensions, the correct stopping rule was the elimination of interior vertices. In d dimensions, we show that the *elimination of interior* $(d-3)$*-dimensional faces* is a good stopping rule. The analysis of the total fragmentation caused by this stopping rule as well as the construction of BSPs in terminal cells requires a better understanding of higher-dimensional geometry. In this section, we establish two key facts about the d-dimensional subdivisions that are central to our analysis.

Lemma 4.1. *Consider two boxes B_1, B_2 that are rods for their container box R in d-space. Let $D_1, D_2 \subset \{1, 2, \ldots, d\}$ be the sets of dimensions in which B_1 and B_2, respectively, are pass-through. If $D_1 \cup D_2 = \{1, 2, \ldots, d\}$, then B_1 and B_2 have intersecting interiors.*

Proof. Let ℓ_i be the extent of dimension x_i *common* to B_1 and B_2. Since in each dimension, at least one of the rods is pass-through, ℓ_i is non-empty for every i. Thus a full-dimensional box defined by the cross-product $\ell_1 \times \cdots \times \ell_d$ lies in the common interior of B_1 and B_2. □

This lemma implies that, for any two boxes B_i, B_j in a subdivision, there is at least one direction in which neither is a rod. With this basic property, we next classify the set system of all possible orientations of $(d-2)$-rods in a box subdivision. The $(d-2)$-rods are the rods especially important to us because our ultimate goal is to consider subdivisions without any interior $(d-3)$-faces. Recall that a set system is a pair $(X, \mathcal{F})$, where X is a ground set, and $\mathcal{F}$ is a family of subsets of X. We consider the set system $(D, \mathcal{F})$, where $D = \{1, 2, \ldots, d\}$, and $|F| = d-2$ for all $F \in \mathcal{F}$. We are interested in the following two configurations:

- **Cycle Configuration:** $\mathcal{F}$ is a *cycle configuration* if $\mathcal{F}$ has at least 3 sets and there is some $x \in D$ that is not an element of any set of $\mathcal{F}$.

- **Star Configuration:** $\mathcal{F}$ is a *star configuration* if $\mathcal{F}$ has at most three sets and they all share a common $(d-3)$ element subset of D.

Lemma 4.2. *Consider a set system $(D, \mathcal{F})$, with $D = \{1, 2, \ldots, d\}$ and $|F| = d-2$ for every $F \in \mathcal{F}$. If every two sets $F_1, F_2 \in \mathcal{F}$ have $F_1 \cup F_2 \neq D$, then $\mathcal{F}$ is either a star or a cycle configuration.*

Proof. Consider the set system $G = (D, E)$, $E = \{D \setminus F : F \in \mathcal{F}\}$. Every $e \in E$ has two elements, and so G is a graph with vertex set D and edge set E. The condition that $F_1 \cup F_2 \neq D$ for any $F_1, F_2 \in \mathcal{F}$ is equivalent to saying that any two distinct edges of E are adjacent. $\mathcal{F}$ is a cycle configuration if and only if G is a star graph (all edges are incident to a common vertex) with at least three edges. $\mathcal{F}$ is a star configuration if and only if G is a subgraph of a triangle.

To prove that $\mathcal{F}$ is either a cycle or a star configuration, it suffices to show that every graph with pairwise adjacent edges is either a triangle or a star graph. Elementary graph theory completes the proof: If $|E| < 3$ then G is trivially a star graph, so let us assume that $|E| \geq 3$. If there is a triangle in G, then $|E| = 3$ and E forms a triangle because any fourth edge would be non-adjacent to at least one edge of the triangle. Now assume that G is triangle-free and consider two adjacent edges, $e_1 = \{a, b\}$ and $e_2 = \{a, c\}$, $b \neq c$. Since any $e_3 \in E \setminus \{e_1, e_2\}$ is adjacent to both e_1 and e_2, but $e_3 \neq \{b, c\}$, e_3 must be incident to a. So every edge is incident to a, and G is a star graph. □

This lemma turns out to be a critical technical piece in constructing linear size BSPs in higher dimensions whenever the subdivision is free of $(d-3)$-dimensional faces. A 4-dimensional version of our star-cycle property was observed by Dumitrescu, Mitchell, and Sharir [8].

5 An Upper Bound in $\mathbb{R}^d$

We begin by showing that if a d-dimensional subdivision is free of all interior $(d-3)$-faces, then it admits a linear size BSP. This fact ensures that if we use the stopping rule "recurse until no interior $(d-3)$-dimensional faces remain," the second phase of the algorithm incurs only a linear additional fragmentation.

Lemma 5.1. *Consider a set of boxes $S = \{B_1, B_2, \ldots, B_n\}$ that forms a subdivision of a d-dimensional container box R. If none of the $(d-3)$-faces of any box B_i is in the interior of R, then there is a BSP of size at most $2n-1$ for S.*

Proof. Our proof is by induction on n. The base case is $n = 1$, for which there is nothing to prove. If any of the boxes of S is a $(d-1)$-rod, then we can make a free cut along a facet of this box by Lemma 2.1; each of the subproblems on either side of the cut contains strictly fewer than n boxes, so our lemma follows by induction. Lemma 2.2 tells us that all boxes of S are $(d-2)$-rods. By Lemmas 4.1 and 4.2, the orientations of these $(d-2)$-rods form either a star or a cycle configuration. We handle these two cases separately.

First, suppose that the orientations of the rods form a cycle configuration. Without loss of generality, let us assume that none of the $(d-2)$-rods is pass-through in direction x_d. We will show that there is a free cut orthogonal to the x_d-axis, and the rest of the proof then follows by induction. We observe that all the $(d-2)$-rods properly intersecting a hyperplane H orthogonal to x_d must have the same orientation; otherwise Lemma 4.1 is violated for the $(d-1)$-dimensional subdivision on H. Now consider a point p on the common boundary of two $(d-2)$-rods of different orientations, and let H be a hyperplane passing through p and orthogonal to x_d. Any (closed) box of S intersecting H lies on one side of H because there are $(d-2)$-rods of different orientations on the two sides of H. Every box intersecting H has a facet along H, and therefore H is a free cut.

Next, suppose that the orientations of the rods form a star configuration. Without loss of generality, assume that all boxes are pass-through in $d-3$ dimensions $x_4, x_5, \ldots, x_d$. We project the boxes of S onto the subspace spanned by the remaining three axes, using the map $\pi : (x_1, x_2, \ldots, x_d) \to (x_1, x_2, x_3)$. Any BSP for the projection corresponds to a BSP of equal size for S, since the pre-image of a plane h in the projection is a hyperplane H in $\mathbb{R}^d$, and H cuts only those $(d-2)$-rods that are the pre-images of the 3-dimensional boxes cut by h. Observe that the subdivision $\pi(S)$ does not have any interior vertices (interior in $\mathbb{R}^3$), because an interior vertex v corresponds to an interior $(d-3)$-face in S, namely, $\pi^{-1}(v)$. By Lemma 3.4, there is a BSP of size at most $2n-1$ for the projected subdivision, which when lifted to $\mathbb{R}^d$ gives a BSP of the same size for S. □

We are now ready to describe our BSP algorithm for d-dimensional subdivisions. In three dimensions, our round robin cuts were made along planes that evenly partition the set of interior vertices. In d dimensions, we choose hyperplanes that "evenly split" the interior $(d-3)$-dimensional faces. Unfortunately, these median hyperplanes can also introduce new $(d-3)$-dimensional faces by splitting some old ones. For instance, in 4-space, a cutting hyperplane can split many 1-faces (line segments). Our fragmentation lemma (Lemma 5.2) below will bound the total number of box fragments created in the round robin phase. We need to introduce a bit of notation first to discuss the objects used in the algorithm.

Given a d-dimensional box B, let $f_k(B)$ denote the set of k-faces of B, where $0 \le k \le d$. We use the notation $f_k(S)$ to refer to the (multi) set of k-faces in all the boxes of $S = \{B_1, B_2, \ldots, B_n\}$; that is, $f_k(S)$ contains multiple copies of a face if that face is incident to multiple boxes. Finally, given a container box R, we use the notation $f_k(R, S)$ to denote the (fragments of) interior k-faces of $f_k(S)$ (faces that intersect the interior of R). In our d-dimensional BSP scheme, the elements of the set $f_{d-3}(R,S)$ will play the rôle that vertices played in Algorithm 3-BSP.

Algorithm *d*-BSP

- Input is a d-dimensional box subdivision $S = \{B_1, B_2, \ldots, B_n\}$ of a container box R.
- Initialize $j = 0$, and $\mathcal{C}_0 = \{R\}$.
- While there is a container box $C \in \mathcal{C}_j$ with non-empty $f_{d-3}(C,S)$, do
  1. For each container box $C \in \mathcal{C}_j$ with non-empty $f_{d-3}(C,S)$, do
     (a) Set $\mathcal{D}_1 = \{C\}$ and let V be the multiset of vertices of all faces in $f_{d-3}(C,S)$.
     (b) For $i = 1$ to d, do
        i. Split every $D \in \mathcal{D}_i$ by the median x_i hyperplane of the multiset $V \cap D$.
        ii. Apply all possible free cuts.
        iii. Let $\mathcal{D}_{i+1}$ be the set of all container boxes resulting from these cuts.
        iv. Remove from V all vertices on the splitting planes used in the first two steps.
  2. Let $\mathcal{C}_{j+1}$ be the set of all container boxes resulting from these cuts, and set $j := j + 1$.
- Return $\mathcal{C}_j$.

Algorithm d-BSP constructs a partition tree T whose leaves correspond to regions that do not contain any fragment of a $(d-3)$-face of the subdivision S. By Lemma 5.1, each of these regions can be refined into a BSP of size linear in the number of box fragments contained in it. Thus, it suffices to bound the total complexity of all the box fragments produced by Algorithm d-BSP.

Lemma 5.2. *Given a d-dimensional box subdivision with n boxes, Algorithm d-BSP produces* $O(n^{\frac{d+1}{3}})$ *box fragments.*

Proof. Let us define $m = |f_{d-3}(R,S)|$. We call a *round* of the algorithm the work done between increments of j. First we show that Algorithm d-BSP terminates in $\lceil(\log m)/3\rceil$ rounds, and then we bound the number of box fragments produced in $\lceil(\log m)/3\rceil$ rounds.

During a round, every $(d-3)$-face F of a subproblem (C,S) can be cut into at most 2^{d-3} fragments—it is cut at most once for each of the $(d-3)$ directions orthogonal to F. We focus on the fragments of F that are not contained in any splitting hyperplane of the round. We argue that at least one vertex of every fragment of F is a vertex of F that does not lie on any of the splitting hyperplanes of the round. A fragment is itself a $(d-3)$-face, and so it is the cross product of 3 fixed coordinates and $d-3$ nontrivial intervals. Each nontrivial interval contains at least one of the two endpoints of the corresponding original interval of F (because there is at most one cut orthogonal to the interval's direction). The cross product of the 3 fixed coordinates and the $d-3$ original interval endpoints is a vertex of F but does not lie on any splitting hyperplane of the round.

The container box C is divided into sub-boxes during the round, each containing at most $2^{d-3} \cdot |f_{d-3}(C,S)|/2^d = |f_{d-3}(C,S)|/8$ vertices of V. As noted, each face fragment produced by the round is incident to a vertex from V, and so $|f_{d-3}(C',S)| \leq |f_{d-3}(C,S)|/8$ for each sub-container C' produced. By induction, the number of interior $(d-3)$-faces at the beginning of round j is at most $m/8^j$. Therefore, after $\lceil \log_8 m\rceil = \lceil(\log m)/3\rceil$ rounds, no $(d-3)$-face intersects the interior of any subproblem's container box, and the algorithm terminates.

Now consider a $(d-2)$-dimensional face of a box B. During a round of the algorithm, it can be cut recursively in $d-2$ steps. Thus, the total number of fragments of $(d-2)$-faces of $f_{d-2}(R,S)$ at round j is $O(n2^{(d-2)j})$.

Finally, in each round j when a box fragment B in some subproblem is split, the container box for the subproblem must have a fragment of a $(d-2)$-face of B in its interior. This is because only a d-rod or a $(d-1)$-rod can have all its $(d-2)$-faces on the boundary of the container by Lemma 2.2. But a d-rod is never split and step 1(b)ii eliminates all $(d-1)$-rods. This implies that $O(n2^{(d-2)j})$ box fragments can be further split in round j. The algorithm terminates in $\lceil(\log m)/3\rceil$ rounds, and so the number of box fragments produced by the algorithm is

$$O\Bigg(n \cdot \sum_{j\leq\lceil\log m/3\rceil} 2^{(d-2)j}\Bigg) \;=\; O\left(n \cdot 2^{(d-2)(\log m)/3}\right) \;=\; O\left(nm^{(d-2)/3}\right) \;=\; O\left(n^{(d+1)/3}\right).$$

□

Since m is at most $2^3\binom{d}{3}\cdot n$, our bound in terms of both n and d is $O(d^{d-2}\cdot n^{(d+1)/3})$.

Finally, this result, combined with Lemma 5.1 and standard implementation techniques similar to those used for Algorithm 3-BSP, yields our main theorem.

Theorem 5.3. *A box subdivision of n boxes in d-space, $d \geq 2$, has a BSP of size $O(n^{\frac{d+1}{3}})$. The BSP can be constructed in time $O(\min(n^{\frac{d+1}{3}}, K\log n))$, where K is the size of the BSP.*

A simple but practical corollary of this theorem is the following: Any collection of n disjoint boxes in d-space whose complement space can be tiled with m additional boxes admits a BSP of size $O((n+m)^{\frac{d+1}{3}})$.

If we perform Algorithm d-BSP with $(d-2)$-faces instead of $(d-3)$-faces (i.e., replacing $f_{d-3}(C,S)$ by $f_{d-2}(C,S)$ everywhere in the algorithm description), then we obtain a BSP for general non-overlapping boxes in d-space: in the resulting partition, no $(d-2)$-face of any input box intersects the interior of any container box, so every such container can be refined to a proper BSP by free cuts. Repeating Lemma 5.2 using $(d-2)$-faces instead of $(d-3)$-faces implies that the modified algorithm terminates in $\lceil(\log m)/2\rceil$ rounds, where $m = |f_{d-2}(R,S)| \leq 2^2\binom{d}{2}\cdot n$. By analogous computation, the total number of box fragments is $O\left(n\cdot 2^{(d-2)(\log m)/2}\right) = O\left(nm^{(d-2)/2}\right) = O\left(n^{d/2}\right)$. This is the best currently known upper bound on the size of a BSP for n non-overlapping general boxes in $\mathbb{R}^d$. We can summarize this argument in the following theorem.

Theorem 5.4. *Every set of n non-overlapping boxes in d-space, $d \geq 2$, has a BSP of size $O(n^{d/2})$.*

One interpretation of our results suggests that for a set S of general axis-aligned boxes in $\mathbb{R}^d$, a round-robin BSP for their $(d-2)$-dimensional faces is also a BSP for S; while for a d-dimensional box subdivision S, a round-robin BSP for only the $(d-3)$-dimensional faces gives a BSP for the input S.

6 Lower Bounds

Paterson and Yao gave a configuration of $3n$ (non-space-filling) axis-aligned rods such that any BSP for the configuration cuts the rods into $\Omega(n^{3/2})$ subcells [15]. (According to Eppstein [9], "Paterson and Yao credit this application of the shape to a personal communication by W. P. Thurston, but the shape itself has been seen before e.g. in Alan Holden's book *Shapes, Space, and Symmetry* (Columbia Univ. Press 1971), p. 161.") We describe a variant of the Paterson-Yao construction in some detail, because it is the basis of our lower bound for box subdivisions.

The rods of the Paterson-Yao construction belong to three families, each parallel to one of the coordinate axes, and form an interlocking grid. See Figure 5. Without loss of generality, assume that $n = \ell^2$ for some integer ℓ. The x-, y-, and z-parallel families consist of boxes indexed by $1 \le i, j, k \le \ell$:

$$\begin{aligned} R_1 &= \{r_1(j,k) = [0, 2\ell] \times [2j, 2j+1] \times [2k, 2k+1] : j, k = 0, 1, \ldots, \ell - 1\}, \\ R_2 &= \{r_2(i,k) = [2i, 2i+1] \times [0, 2\ell] \times [2k+1, 2k+2] : i, k = 0, 1, \ldots, \ell - 1\}, \\ R_3 &= \{r_3(i,j) = [2i+1, 2i+2] \times [2j+1, 2j+2] \times [0, 2\ell] : i, j = 0, 1, \ldots, \ell - 1\}. \end{aligned}$$

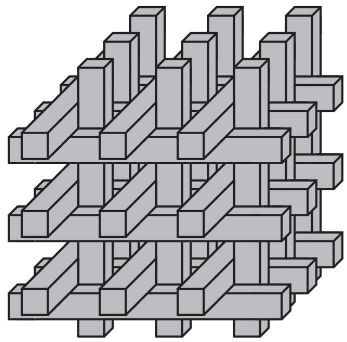

Figure 5: The Paterson-Yao lower bound construction, with the rods slightly separated and lengthened to make them easier to see.

It is straightforward to observe that the rods in each family are disjoint. Furthermore, rods in different families are disjoint, because for each pair of families there is one of the three dimensions (x, y, or z) in which the two families lie in disjoint ranges: for one family, any rod's coordinates lie in the range $[2A, 2A+1]$, for some integer A; for the other family, any rod's coordinates lie in $[2B+1, 2B+2]$, for integral B. Note also that each grid point $(2i+1, 2j+1, 2k+1)$, for $1 \le i, j, k \le \ell$, is incident to one rod from each of the three families.

For every $i, j, k = 0, 1, \ldots, \ell - 1$, consider the box

$$[2i, 2i+2] \times [2j, 2j+2] \times [2k, 2k+2].$$

We call these boxes *junctions*. The junctions are interior-disjoint boxes, and each must be cut by a BSP plane passing through its center, since the BSP must separate the three rods in the neighborhood of the center $(2i+1, 2j+1, 2k+1)$. The first plane that passes through the junction must cut at least one of the three rods incident to the center. Since all the junctions are disjoint, the total number of rod cuts is at least $\ell^3 = n^{3/2}$.

The Paterson-Yao construction is *opaque*, in the sense that every axis-parallel line passing through the container cube

$$[0, 2\ell] \times [0, 2\ell] \times [0, 2\ell]$$

intersects the closure of at least one rod. However, the configuration of rods is not a subdivision: the rods do not fill space. This is easy to prove by observing that the total volume of the container cube is $8\ell^3$, and the total volume of each rod inside is 2ℓ. There are $3\ell^2$ rods, for a total rod volume inside the container cube of $6\ell^3$, which is less than $8\ell^3$. The empty space is distributed in $2\ell^3$ unit cubes. In particular, for each junction for i, j, k, the unit cubes

$$[2i, 2i+1] \times [2j+1, 2j+2] \times [2k, 2k+1]$$

and

$$[2i+1, 2i+2] \times [2j, 2j+1] \times [2k+1, 2k+2]$$

are empty. See Figure 6.

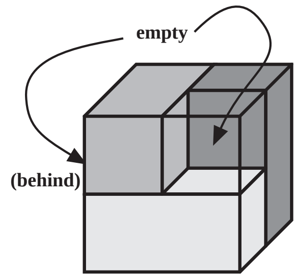

Figure 6: A unit cube centered on a grid point in the Paterson-Yao construction.

6.1 Lower Bound Construction for 3D Subdivisions

Converting the Paterson-Yao configuration into a subdivision requires adding $\Theta(\ell^3)$ new cells—two unit cubes for each junction. Thus the $\Omega(\ell^3)$ lower bound on the BSP complexity becomes trivial—it is linear in the input size. However, with a little more work, we can extend the construction to give a nontrivial lower bound.

Theorem 6.1. *There is a three-dimensional subdivision of space into n axis-aligned boxes such that any axis-aligned BSP for the subdivision cuts the boxes into $\Omega(n^{4/3})$ subcells.*

Proof. We begin with a Paterson-Yao construction consisting of $3\ell^2$ rods, and then extend it to a subdivision of the container cube $[0, 2\ell]^3$ by adding the $2\ell^3$ unit cubes needed to fill in the empty space between the rods. We then subdivide each rod longitudinally into ℓ parallel sub-rods.

As in the original Paterson-Yao argument, each of the ℓ^3 junctions must be cut by at least one BSP plane, since the BSP must separate the cells incident to $(2i+1, 2j+1, 2k+1)$. The first plane that cuts a junction must completely cross one of the original rods, and hence it cuts at least ℓ sub-rods. The total number of rod cuts is at least $\ell^3 \times \ell = \ell^4$. The total number of cells in our subdivision is $n = 3\ell^2 \times \ell + 2\ell^3 = 5\ell^3$, and hence the number of subcells produced by the BSP is $\Omega(n^{4/3})$. □

6.2 Constructions for Multidimensional Subdivisions

Our multidimensional construction is similar to in spirit to the previous construction. This construction is recursive—we use the lower bound construction in $(d-1)$-space to build the subdivision in d dimensions. We begin with an informal description, then formally describe the construction in four dimensions. The construction in d dimensions is a straightforward extension of this construction.

We partition the container box in a grid-like fashion into disjoint regions that we call *junctions*, and give a lower bound for the BSP complexity for the fragments of objects within each region. Then we bound the complexity of a BSP for all objects by the sum of complexities over all the regions.

We have $d-1$ families of congruent $(d-2)$-rods whose orientations form a cycle configuration, and a family of 1-rods whose orientation is the complement of the orientations of the $(d-2)$-rods. The rods ensure our complexity bound in each region. (Intuitively, it is efficient to use rods of larger orientations, because a k-rod of the container box is a k-rod in each junction it intersects.) On the other hand, rods of such orientations cannot fill the container box, and therefore we need to add a number of filler boxes to obtain a box subdivision.

In order to balance the number of rods and the number of filler boxes, we allocate the rods in *bundles*, so that the union of a bundle of rods is again a rod with the same orientation. The

$(d-2)$-rods in a bundle are all congruent; the $\mathbb{R}^{d-1}$-projection of a bundle of 1-rods, however, is the worst-case box subdivision we obtain in $\mathbb{R}^{d-1}$.

Theorem 6.2. *There is a four-dimensional box subdivision with n boxes such that the size of any axis-aligned BSP is $\Omega(n^{13/9})$.*

Proof. First, we describe an auxiliary construction of $\Theta(\ell^{9/2})$ boxes such that all vertices of the boxes lie on a $2\ell \times 2\ell \times 2\ell \times 3\ell^{3/2}$ grid. The rods of the auxiliary subdivision correspond to the bundles of rods in the final subdivision.

We have ℓ^3 long 1-rods that are pass-through in x_4:

$$R_1 = \{r_1(i,j,k) = [2i, 2i+1] \times [2j, 2j+1] \times [2k, 2k+1] \times [0, 3\ell^{3/2}] : i,j,k = 0,\ldots,\ell-1\}.$$

We also have three families of boxes, each containing $\ell^{5/2}$ 2-rods with orientations $\{x_1, x_2\}$, $\{x_1, x_3\}$, and $\{x_2, x_3\}$, respectively:

$$R_{1,2} = \{r_{1,2}(k,t) = [0, 2\ell] \times [0, 2\ell] \times [2k+1, 2k+2] \times [3t, 3t+1] : k = 0,1,\ldots,\ell-1;\ t = 0,\ldots,\ell^{3/2}-1\},$$

$$R_{1,3} = \{r_{1,3}(j,t) = [0, 2\ell] \times [2j+1, 2j+2] \times [0, 2\ell] \times [3t+1, 3t+2] : j = 0,\ldots,\ell-1;\ t = 0,\ldots,\ell^{3/2}-1\},$$

$$R_{2,3} = \{r_{2,3}(i,t) = [2i+1, 2i+2] \times [0, 2\ell] \times [0, 2\ell] \times [3t+2, 3t+3] : i = 0,\ldots,\ell-1;\ t = 0,\ldots,\ell^{3/2}-1\}.$$

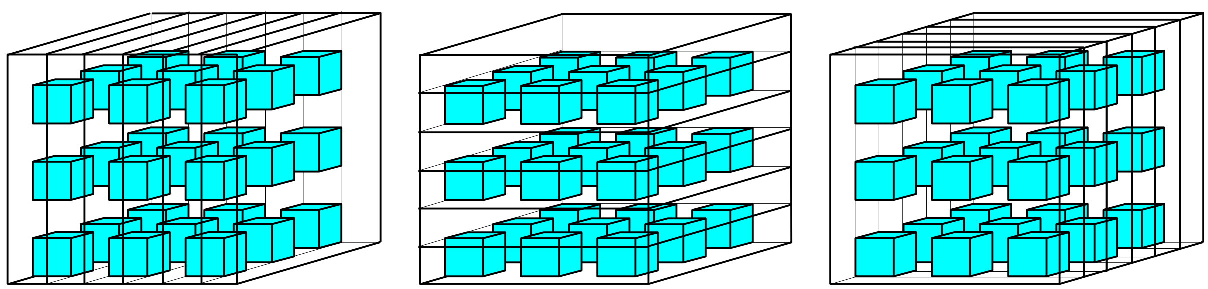

Figure 7: Three different 3-dimensional slices of the subdivision along three hyperplanes orthogonal to x_4.

Figure 7 illustrates the intersection of our auxiliary construction with three (3-dimensional) hyperplanes orthogonal to x_4. The intersection of the 1-rods in R_1 with every hyperplane consists of ℓ^3 unit cubes in a grid. The 2-rods of three distinct orientations appear in three different slices. In each slice, congruent 2-rods fill ℓ slots between the grid of unit cubes.

Since all the rods have pairwise disjoint interiors, we can obtain an orthogonal subdivision of the container box $R = [0, 2\ell] \times [0, 2\ell] \times [0, 2\ell] \times [0, 3\ell^{3/2}]$ by filling up the space between the rods with $9\ell^{9/2}$ unit cubes. This completes the description of our auxiliary construction of $\Theta(\ell^{9/2})$ boxes. We obtain a subdivision S of $n = \Theta(\ell^{9/2})$ boxes in the following way: partition each 2-rod into ℓ^2 congruent 2-rods with the same orientation, and replace each 1-rod by $\ell^{3/2}$ pairwise disjoint 1-rods whose projection to the (x_2, x_3, x_4) hyperplane is the worst-case construction for a subdivision in $\mathbb{R}^3$.

Consider the $\ell^{9/2}$ boxes $\Pi(i,j,k,t) = [2i, 2i+2] \times [2j, 2j+2] \times [2k, 2k+2] \times [3t, 3t+3]$, where $i,j,k = 0,1,\ldots,\ell-1$ and $t = 0,1,\ldots,\ell^{3/2}-1$. These are the *junction* regions. Note that the junctions have pairwise disjoint interiors. We show that any axis-aligned BSP for S dissects the rods of S such that pieces of rods have at least ℓ^2 vertices in the interior of every junction. This implies that the total number of pieces is $\Omega(\ell^{9/2} \cdot \ell^2) = \Omega(\ell^{13/2}) = \Omega(n^{13/9})$.

First suppose that each of the $\ell^{3/2}$ 1-rods in $r_1(i,j,k) \in R_1$ is cut by some hyperplane $x_4 = c$, for $c \in (3t, 3t+3)$. (Not all 1-rods might be cut by the same hyperplane, though.) Consider an intersection of the bundle of 1-rods with a hyperplane $x_4 = c_0$ such that each 1-rod has a vertex in the interior of $\Pi(i,j,k,t)$ below $x_4 = c_0$. A BSP for S, restricted to the intersection of the bundle with $x_4 = c_0$, must be a BSP for the 3-dimensional subdivision to which the bundle projects. Hence the intersection BSP must have $\Omega((\ell^{3/2})^{4/3}) = \Omega(\ell^2)$ vertices, by the previous lower bound. Each vertex corresponds to a vertex in the interior of $\Pi(i,j,k,t)$—in fact, it is connected to its corresponding vertex by an edge of the BSP parallel to the x_4-axis.

Otherwise let r be a rod in $R_1(i,j,k)$ that is not cut by any hyperplane $x_4 = c$, for $c \in (3t, 3t+3)$. Let e be a segment in the interior of r such that e is pass-through in x_4 within $\Pi(i,j,k,t)$ and does not lie on any hyperplane of the BSP. Since e is not cut by the BSP, one subproblem contains e all through the partitioning procedure. Notice that the container box for e cannot be separated from all the other original boxes by hyperplanes orthogonal to only one axis. Let H_1 and H_2 be hyperplanes orthogonal to, say, x_1 and x_2 on the boundary of the final container box of e. The 2-face $F = H_1 \cap H_2$ is pass-through in x_3 and x_4 and lies in the interior of $\Pi(i,j,k,t)$. Therefore F has a common vertex with all ℓ^2 2-rods in $r_{1,2}(k,t)$ in the interior of $\Pi(i,j,k,t)$. $\square$

Generalizing this recursive construction, we obtain a recursive formula for the exponent of the lower bound $\beta(d)$ we can reach for a box subdivision in $\mathbb{R}^d$.

Theorem 6.3. *There is a d-dimensional box subdivision S with n boxes such that the size of any axis-aligned BSP for S is $\Omega(n^{\beta(d)})$, where $\beta(3) = 4/3$, $\beta(4) = 13/9$, and $\beta(d)$ converges to $(1+\sqrt{5})/2$ as $d \to \infty$.*

Proof. In $\mathbb{R}^d$, our auxiliary construction consists of ℓ^{d-1} 1-rods that are pass-through in x_d, and $d-1$ families of $\ell^{1+\alpha(d)}$ $(d-2)$-rods such that each slice $x_d = c$ intersects ℓ $(d-2)$-rods and there are $\ell^{\alpha(d)}$ levels in the x_d extent ($\alpha(d)$ is a parameter to be specified later). We need $\Theta(\ell^{(d-1)+\alpha(d)})$ unit cubes to convert this configuration into a subdivision.

We balance the number of filler unit cubes and rods by replacing each auxiliary $(d-2)$-rod by a bundle of ℓ^{d-2} congruent $(d-2)$-rods and each auxiliary 1-rod by a bundle of $\ell^{\alpha(d)}$ 1-rods. We arrange the 1-rods in each bundle such that their $(d-1)$-dimensional projection gives the $\mathbb{R}^{d-1}$ subdivision for which any BSP has size $\Omega((\ell^{\alpha(d)})^{\beta(d-1)})$.

We define $\ell^{(d-1)+\alpha(d)}$ junctions, arranged in a grid. Each junction contains either $\Omega(\ell^{d-2})$ vertices on fragments of $(d-2)$-rods or $\Omega((\ell^{\alpha(d)})^{\beta(d-1)})$ vertices on fragments of 1-rods. To achieve our bound, we set $\alpha(d) = (d-2)/\beta(d-1)$, implying that any BSP for our subdivision has at least $\ell^{(d-1)+\alpha(d)} \cdot \ell^{d-2} = \ell^{2d-3+\alpha(d)}$ vertices. That gives a lower bound of

$$\Omega\left(n^{\frac{2d-3+\alpha(d)}{d-1+\alpha(d)}}\right) = \Omega\left(n^{\frac{2d-3+(d-2)/\beta(d-1)}{d-1+(d-2)/\beta(d-1)}}\right).$$

The exponent in this bound is

$$\beta(d) = \frac{2d-3+(d-2)/\beta(d-1)}{d-1+(d-2)/\beta(d-1)} = 1 + \frac{d-2}{d-1+(d-2)/\beta(d-1)}.$$

The first values of this sequence are $\beta(4) = 13/9 = 1.444$, $\beta(5) = 118/79 = 1.494$, $\beta(6) = 689/453 = 1.521$, and $\beta(7) = 9844/6399 = 1.538$. The sequence converges to $\lim_{d\to\infty} \beta(d) = (1+\sqrt{5})/2 = 1.618$.

Note that the d-dimensional lower bound subsumes the lower bound for three dimensions. If $d = 3$, then $\beta(d-1) = \beta(2) = 1$, and $\alpha(3) = (3-2)/\beta(2) = 1$. The lower bound we obtain is

$$\Omega\left(n^{\frac{2\cdot 3-3+\alpha(3)}{3-1+\alpha(3)}}\right) = \Omega(n^{4/3}),$$

matching Theorem 6.1. □

7 Concluding Remarks

Binary space partitions are a popular space-decomposition method in computational geometry, computer graphics, and other fields dealing with geometric shapes. However, their worst-case complexity remains poorly understood in dimensions three and higher. Even for the BSP of n orthogonal boxes in d dimensions, the best upper bound known is $O(n^{d/2})$, while no lower bound better than $\Omega(n^2)$ is known.

In an attempt to understand the complexity of higher-dimensional BSPs, we considered a natural special case: orthogonal subdivisions. We showed that every d-dimensional subdivision with n boxes admits a BSP of size $O(n^{\frac{d+1}{3}})$. We also exhibited subdivisions in d dimensions for which every axis-aligned BSP must have size $\Omega(n^{\beta(d)})$, where $\beta(d)$ converges to $(1+\sqrt{5})/2$ as $d \to \infty$. Our result shows that if the objects do not fracture the complement space too badly, then their BSP size can be significantly smaller than the current worst-case bounds indicate.

Clearly, the most interesting open problem suggested by this paper is to close the gap between the upper and lower bounds. The absence of super-quadratic lower bounds is intriguing. Can it really be the case that in any dimension d, a set of n boxes (forming a subdivision, or not) admits a quadratic size BSP? If not, then what is the true complexity of the BSP of n d-dimensional boxes? Is this bound necessarily worse than that for n boxes forming a subdivision in higher dimensions?

References

[1] P. K. Agarwal, E. F. Grove, T. M. Murali and J. S. Vitter, Binary space partitions for fat rectangles, *SIAM J. of Computing* **29** (2000), 1422–1448.

[2] C. Ballieux, Motion planning using binary space partitions, Tech. Rep. Inf/src/93-25, Utrecht University, 1993.

[3] J. L. Bentley, Multidimensional binary search trees used for associative searching, *Communication of the ACM* **18** (9) (1975), 509–517.

[4] M. de Berg, Linear size binary space partitions for uncluttered scenes, *Algorithmica* **28** (2000), 353–366.

[5] P. Berman, B. DasGupta, and S. Muthukrishnan, On the exact size of the binary space partitioning of sets of isothetic rectangles with applications, *SIAM J. Discrete Mathematics* **15** (2) (2002), 252–267.

[6] N. Chin and S. Feiner, Near real-time shadow generation using BSP trees, in: *Proc. SIGGRAPH, 1989*, vol. 23, ACM SIGGRAPH, pp. 99–106.

[7] N. Chin and S. Feiner, Fast object-precision shadow generation for areal light sources using BSP trees, *Comput. Graph.* **25** (1992), 21–30, Proc. 1992 Sympos. Interactive 3D Graphics.

[8] A. Dumitrescu, J. S. B. Mitchell, and M. Sharir, Binary space partitions for axis-parallel segments, rectangles, and hyperrectangles, *Discrete Comput, Geom.* **31** (2) (2004), 207–227.

[9] D. Eppstein, Untetrahedralizable objects, *The Geometry Junkyard*, `http://www.ics.uci.edu/~eppstein/junkyard/untetra/`.

[10] H. Fuchs, Z. M. Kedem, and B. Naylor, On visible surface generation by a priori tree structures, *Comput. Graph.* **14** (3) (1980), 124–133, Proc. SIGGRAPH '80.

[11] T. M. Murali, Efficient hidden-surface removal in theory and in practice, Ph. D. Thesis, Department of Computer Science, Brown University, Providence, 1998, pp. 80-81.

[12] B. Naylor, J. A. Amanatides, and W. Thibault, Merging BSP trees yields polyhedral set operations, *Comput. Graph.* **24** (4) (1990), 115–124, Proc. SIGGRAPH '90.

[13] B. Naylor and W. Thibault, Application of BSP trees to ray-tracing and CSG evaluation, Technical Report GIT-ICS 86/03, Georgia Institute of Tech., School of Information and Computer Science, 1986.

[14] M. S. Paterson and F. F. Yao, Efficient binary space partitions for hidden-surface removal and solid modeling, *Discrete Comput. Geom.* **5** (1990), 485–503.

[15] M. S. Paterson and F. F. Yao, Optimal binary space partitions for orthogonal objects, *J. Algorithms* **13** (1992), 99–113.

[16] R. A. Schumacker, R. Brand, M. Gilliland, and W. Sharp, Study for applying computer-generated images to visual simulation, Tech. Rep. AFHRL–TR–69–14, U.S. Air Force Human Resources Laboratory, 1969.

[17] W. C. Thibault and B. F. Naylor, Set operations on polyhedra using binary space partitioning trees, *Comput. Graph.* **21** (4) (1987), 153–162, Proc. SIGGRAPH '87.

[18] Cs. D. Tóth, A note on binary plane partitions, *Discrete Comput. Geom.* **30** (2003), 3–16.

[19] Cs. D. Tóth, Binary space partitions for orthogonal fat rectangles, in: *Proc. 11th European Sympos. Algorithms*, vol. 2832 of LNCS, Springer-Verlag, Berlin, 2003, pp. 494–505.