Market Clearability

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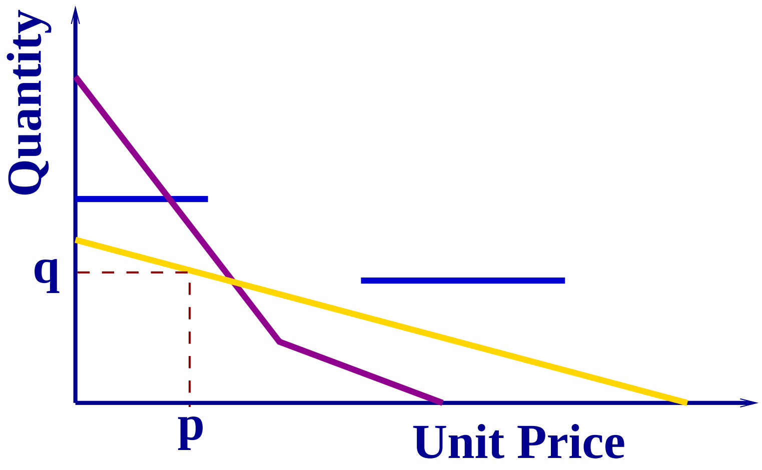
Market Clearing

- Computational complexity of clearing markets.
- Forward and reverse auctions.
- Homogeneous goods: Identical units.
  e.g. electricity, bandwidth, oil, stocks etc.
- Objective is to maximize revenue.
- Complexity depends on divisibility and price discrimination.
  1. Price-quantity pairs vs. curves.
  2. Piecewise linear vs. linear curves.
  3. Uniform vs. discriminatory prices.

Demand Curves

- **Auctioneer has Q identical units to sell.**
- **N buyers, each with a demand curve. Demand at price p is $q(p)$.**

  1. **Arbitrary piecewise linear curves.**
  2. **Price-Quantity pair bid a special case.**
  3. **Linear demand curves.**



- **Pricing can be uniform or discriminatory.**
- **Goal is to maximize revenue $\sum_{i=1}^{n} p_i q_i$.**

Main Results

- **Uniform clearing price and piecewise linear demand curves: Max-revenue auction in $O(N \log N)$ time.**
- **Same problem with discriminatory clearing prices is NP-Complete.**
- **Even with discriminatory prices, if the demand curves are linear, auction solved in $O(N \log N)$ time.**
- **Same holds for the reverse auction.**
- **Max-revenue auction with indivisible price-quantity pair bids is NP-Complete.**
- **Approximation algorithms and extensions to XOR or OR-of-XOR price-quantity bids.**

Related Work

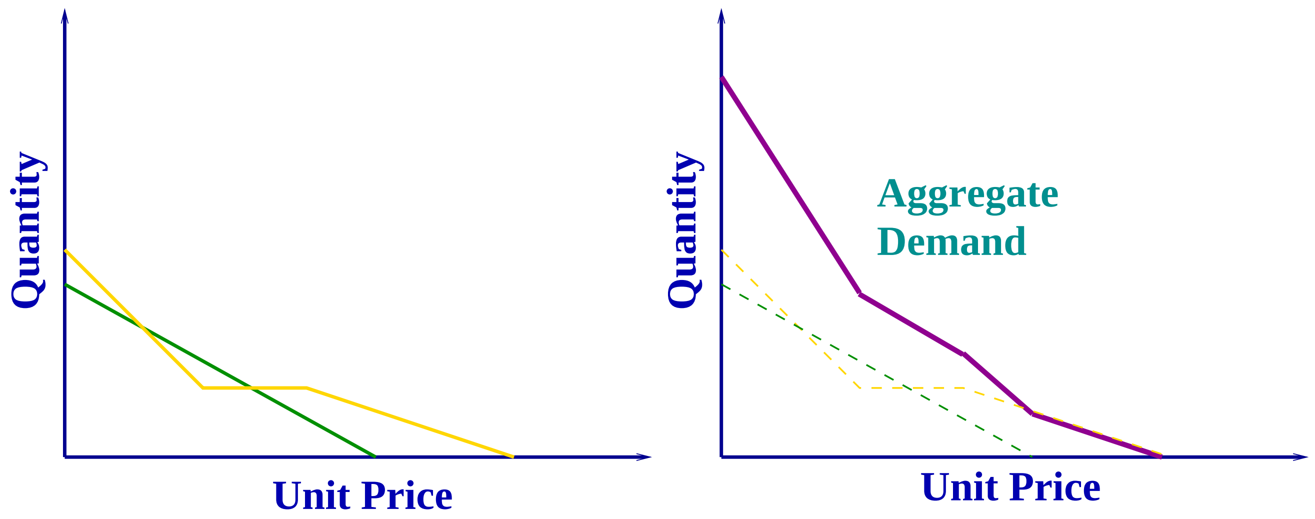
- Economics literature on suppy/demand curves aimed at finding equilibrium.
- Combinatorial auctions too general for our setting.
- Auctions with indivisible price-quantity bids are related to knapsack problem.
- [Wurman-Walsh-Wellman] consider uniform-price market clearing.
  No revenue consideration.
- [Lavi-Nisan] consider online auctions with supply curves. Sub-optimal revenue.
- Our focus is on demand curves, optimal revenue, and computational complexity.

Outline

- **Uniform price auction with piecewise linear curves.**
- **NP-Completeness of discriminatory price auction.**
- **Discriminatory auction with linear curves.**
- **Indivisible price-quantity pair bids.**
- **Conclusions.**

Uniform Price Auction

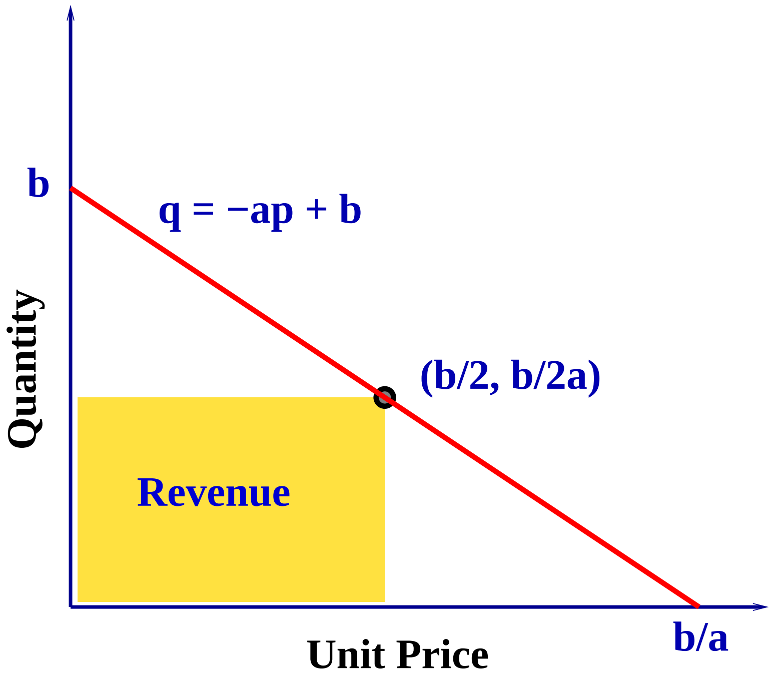
- Compute the aggregate demand curve.
- The aggregate curve is piecewise linear.



- Solve the max-revenue auction for each linear piece independently.
- Suppose unit price p^* maximizes the revenue over all pieces.
- Clear each buyer at p^*.
  Buyer i gets $q_i(p^*)$ units.

Linear Demand Curve

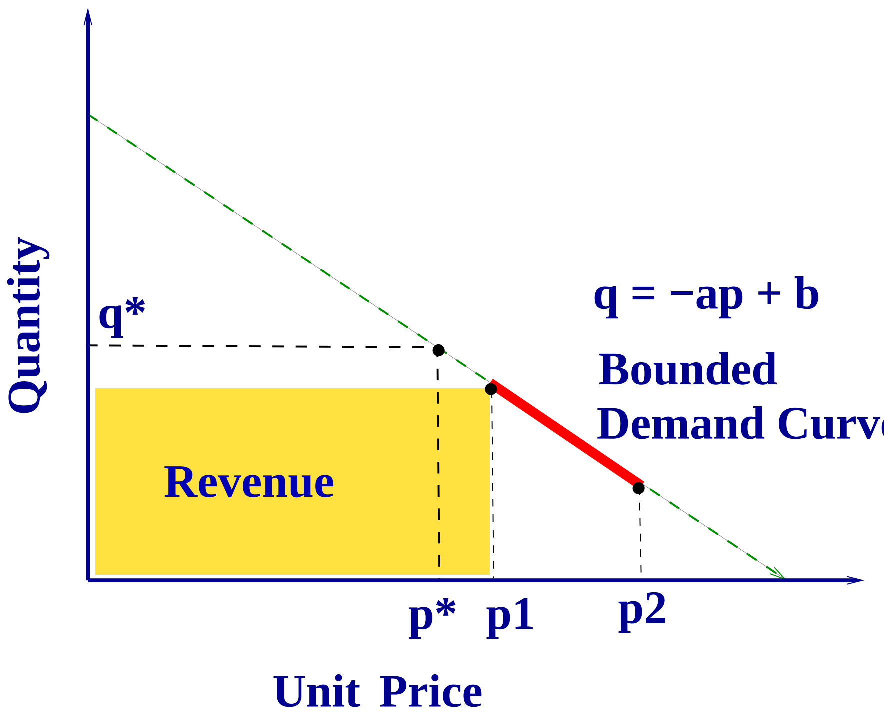
- **Demand curve:** $q = -ap + b$
  Downward sloping for illustration
- **Revenue at price p is $-ap^2 + bp$.**
- **Maximum revenue when** $p^* = \frac{b}{2a}$, $q^* = \frac{b}{2}$.



- **Geometrically, the auction clears at midpoint of the demand curve.**

Bounded Linear Piece

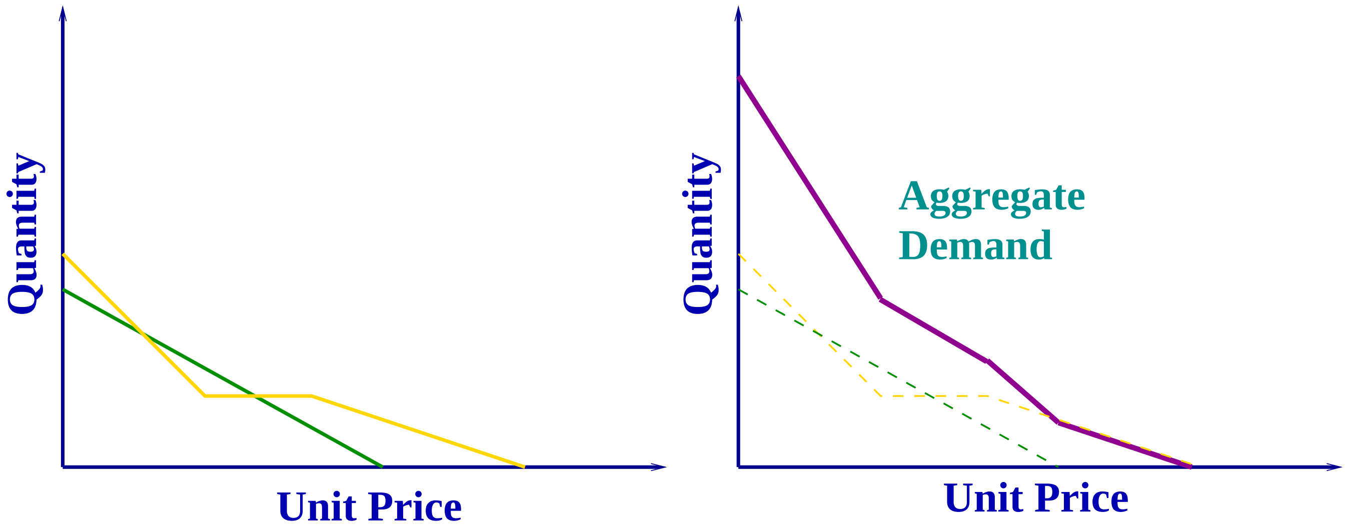
- **Linear curve** $q = -ap + b$ **in range** $[p_1, p_2]$



- **Optimum either at** p^***, or the endpoint closest to** p^***.**
- **Auction solved for a linear piece in** $O(1)$ **time.**

Complexity

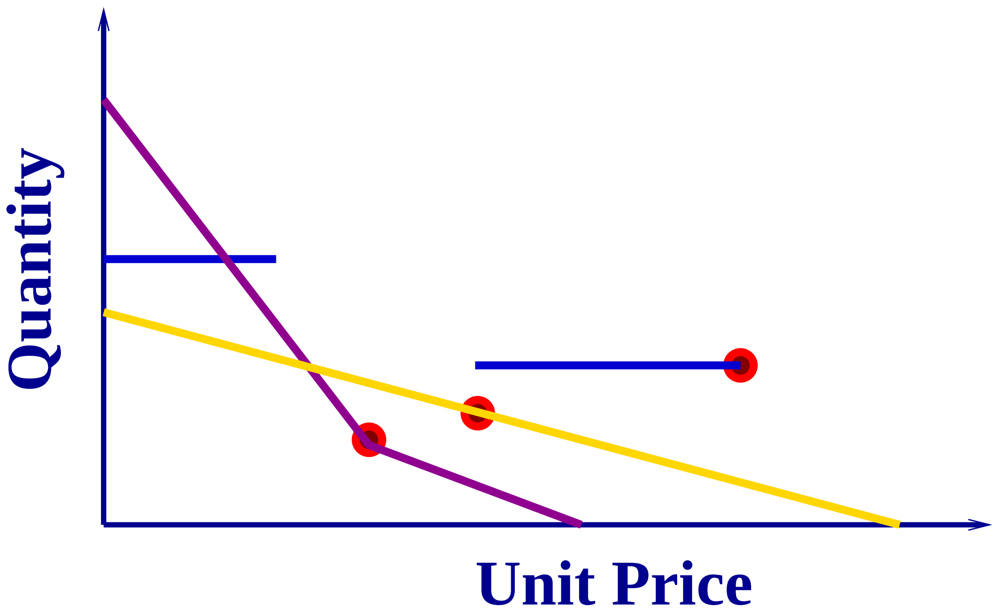
- **Aggregate demand curve computed in $O(K \log K)$ time, if K linear pieces.**



- **For each piece, the auction can be solved in $O(1)$ time.**
- **Revenue-maximizing, uniform price auction with piecewise linear demand curves can be solved in worst-case time $O(K \log K)$.**

Discriminatory Auctions

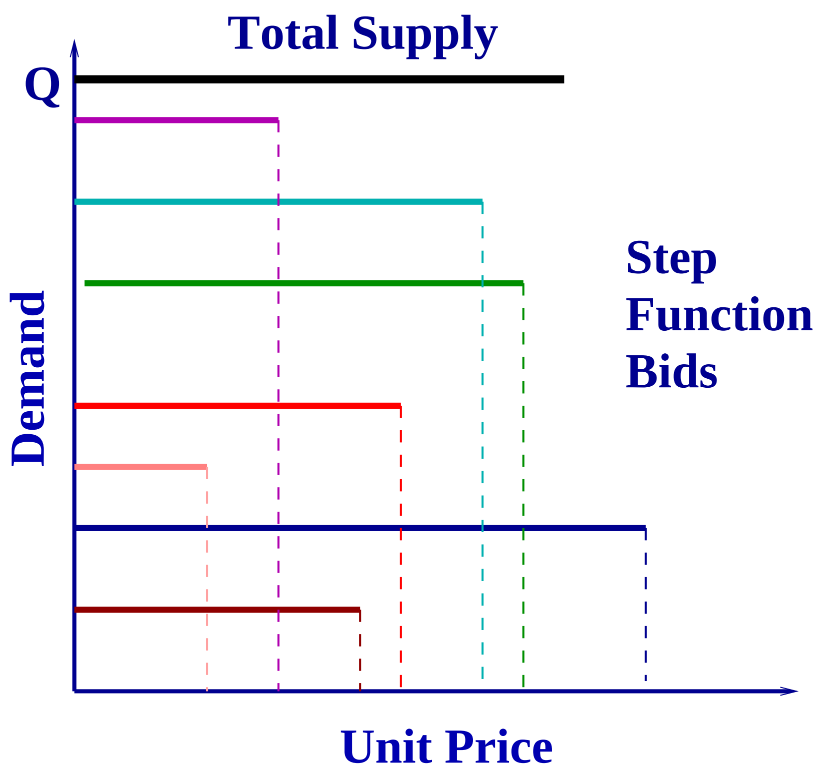
- Buyers may be cleared at different prices.



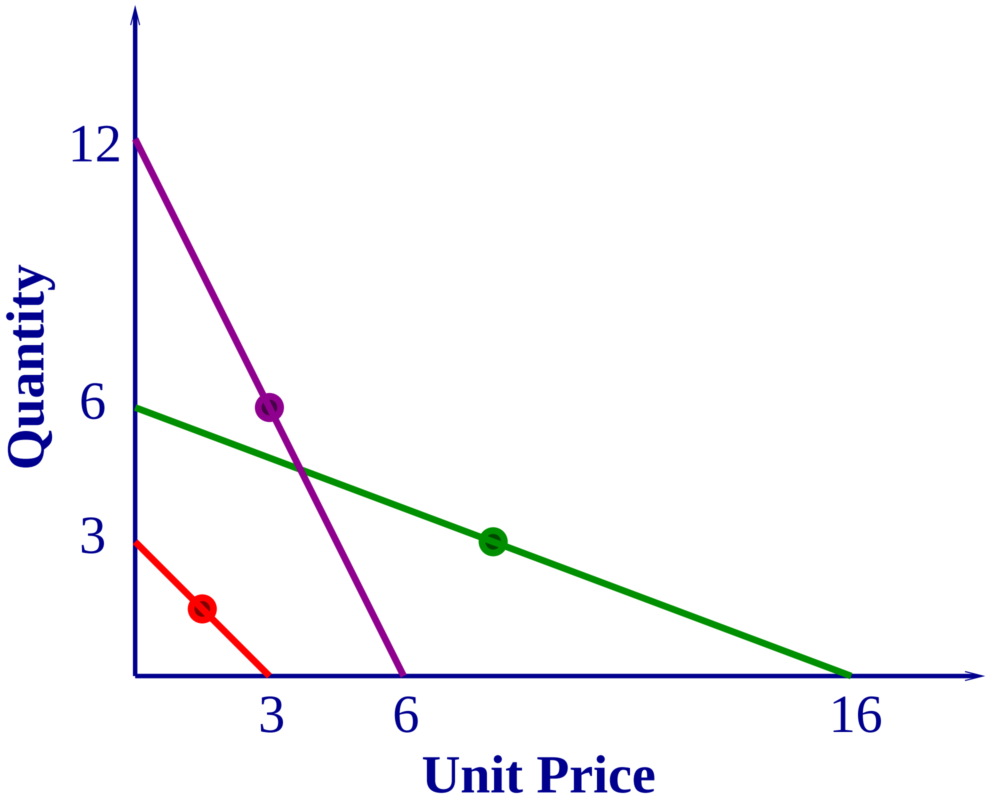
- Maximize $\sum_j p_j q_j$ subject to demand curves and quantity available.
- Discriminatory price auctions lead to potentially higher revenue.
- However, max-revenue auction with discriminatory prices is NP-Complete.

Intractability

- **Reduction from Knapsack Problem.**
- **Item** $(size\ s, val\ v)$ **transformed to step function bid: buy** s **units at total price at most** v**; no units at a higher price.**
- **Knapsack size** Q **is units available.**



Algorithm for Linear Curves



- **Independently determine optimal clearing for each buyer:** (p_j^*, q_j^*)
- **If** $\sum_{j=1}^{n} q_j^* \leq Q$, **this is optimal.**
- **Otherwise** $\cdots$

Lagrangian Solution

- **Maximize** $\sum_{j=1}^{n} p_j q_j$ **s.t.**
  - **Demand:** $q_j = -a_j p_j + b_j$
  - **Supply:** $\sum_j q_j \leq Q$
- **Eliminate p_j's from obj. and add supply constraint using Lagrangian multiplier**

$$\max \quad \left(\frac{b_j q_j}{a_j} - \frac{q_j^2}{a_j} \right) + \lambda (Q - \sum_{j=1}^{n} q_j).$$

- **Solution**

$$q_j = \frac{b_j}{2} - \frac{a_j}{2} \left(\frac{\sum b_j - 2Q}{\sum a_j} \right)$$

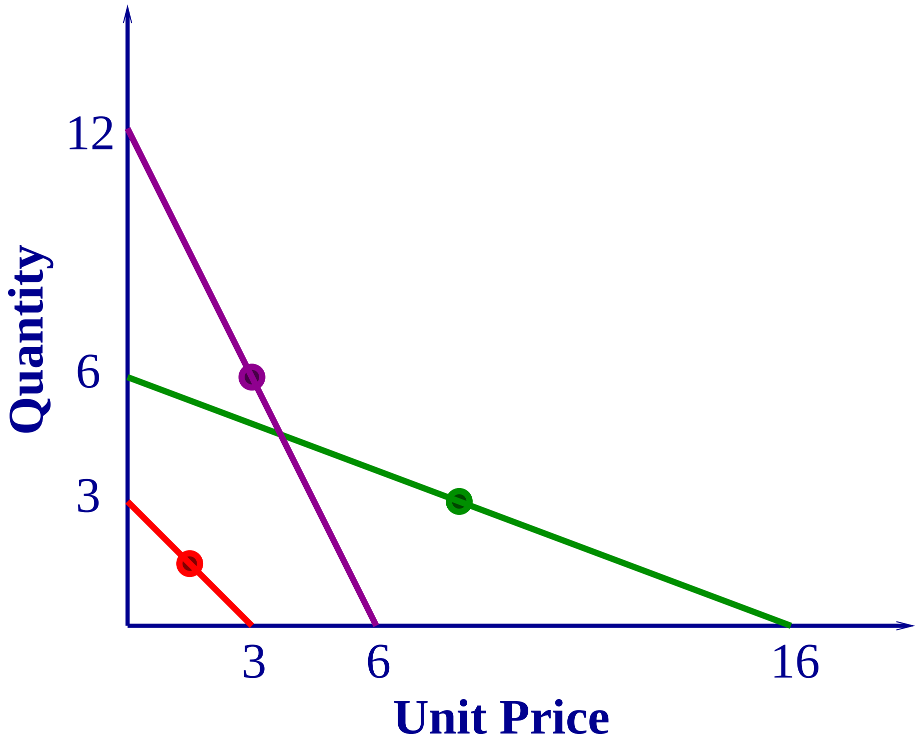
$$p_j = \frac{b_j}{2a_j} + \frac{1}{2} \left(\frac{\sum b_j - 2Q}{\sum a_j} \right)$$

- **Supply constraint raises clearing prices for all buyers uniformly!**

Fly in the Ointment · · ·

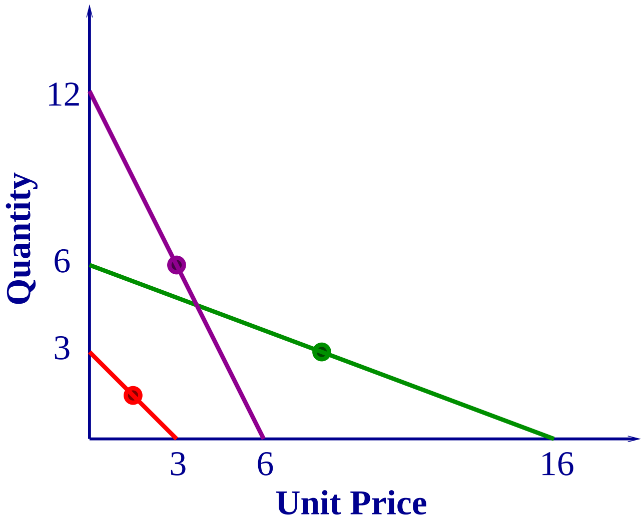
- **But Lagrangian solution is invalid if some quantity q_j is negative.**

$$
\begin{aligned}
q_j &= \frac{b_j}{2} - \frac{a_j}{2}\left(\frac{\sum b_j - 2Q}{\sum a_j}\right) \\
p_j &= \frac{b_j}{2a_j} + \frac{1}{2}\left(\frac{\sum b_j - 2Q}{\sum a_j}\right)
\end{aligned}
$$



Raise-and-Drop Algorithm

1. **Bids** $S = \{1, 2, \ldots, n\}$**.**
2. **Initialize** $(p_j, q_j) = \left(\frac{b_j}{2a_j}, \frac{b_j}{2}\right)$**.**
3. **If** $\sum_{j \in S} q_j \le Q$**, done.**
4. **Let** ℓ **be bid with min** p_j**.**
5. **Set** $p'_j = p_j + p_\ell$**,** $q'_j = -a_j p'_j + b_j$**, for** $j \in S$**.**
6. **If** $\sum_{j \in S} q'_j \le Q$**, output the Lagrangian solution using bids of** S**.**
7. $S = S - \{\ell\}$**, and go to step 4.**



Discriminatory Auction Theorems

Theorem: If demand curves are piecewise linear, then max-revenue auction is NP-Complete.

Theorem: If demands are downward sloping linear curves, then max-revenue auction can be solved in $O(N \log N)$ time.

Price-Quantity Pair Bids

- **A price-quantity pair bid (p, q) is all-or-nothing: bidder pays \$$p$ per unit for q units.**
- **Max-revenue auction with price-quantity pair bids is NP-Complete: reduction from knapsack.**
- **Pseudo-polynomial algorithm of time complexity $O(n^2 V)$, where V is maximum (total) bid price.**
- $(1-\varepsilon)$ **approximation in** $O(n \log \frac{1}{\varepsilon} + \frac{1}{\varepsilon^4})$**.**
- **We extend these results to include XOR or OR-of-XOR bids.**

Closing Remarks

- **Computational complexity of forward and reverse auctions for homogeneous goods.**
- **Max-revenue auctions with uniform prices and piecewise linear demand curves solved in $O(N \log N)$ time.**
- **Same problem with discriminatory prices is NP-Complete.**
- **With linear demand curves, even discriminatory prices auctions can be solved in $O(N \log N)$ time.**
- **Max-revenue auction with price-quantity pair bids is NP-Complete in general.**
- **Pseudo-polynomial algorithms, or polynomial approximations.**

An Example

- **Total supply** $Q = 50$.
- **Unconstrained:** $q_1 = 0.5$, $q_2 = 5$, $q_3 = 50$.
- **Revenue with buyer 3 alone:** $\$2500$.
- **Optimal:** $q_1 = 0$, $q_2 = 2.5$, $q_3 = 47.5$, **with revenue** $\$2512.5$.