Shape Sensitive Geometric Permutations

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Geometric Permutations

- Disjoint convex objects $S$ in $\mathcal{R}^d$.
- $\ell$ a line transversal if it intersects all objects of $S$.
- The order induced by $\ell$ is a geometric permutation. (Line is undirected.)
- Example shows two permutations: $(A, B, C, D)$ and $(A, C, B, D)$.
- How many permutations?
Two Dimensions

- $n$ convex objects in 2D with $2n - 2$ distinct permutations. [KLZ 85]
- $2n - 2$ also an upper bound. [ES 90]
Known Bounds

- $g_d(S)$ is number of permutations for $S$.
- $g_d(n) = \max\{g_d(S) \mid |S| = n\}$.
- The following bounds are known:
  1. $g_d(n) = \Omega(n^{d-1})$ [KLL 92].
  2. $g_d(n) = O(n^{2d-2})$ [W 90].
- Large gap between upper and lower bounds.
- Specialized family of convex objects, balls, [SMS 00].
  1. $g_d(n) = \Theta(n^{d-1})$ for balls in $\mathcal{R}^d$.
  2. For $d = 2$, congruent disks same radius have at most 2 permutations.
- Conjecture: $g_d(n) = O(1)$ for congruent balls.
New Results: Shape Sensitivity

- A set of $n$ unit balls in $\mathcal{R}^d$ admits at most $4$ permutations, with $n$ depending on $d$.

- A set of $n$ arbitrary size, but axis-aligned, rectangular boxes in $\mathcal{R}^d$ admits at most $2^{d-1}$ permutations.

- A matching lower bound of $2^{d-1}$ for boxes (cubes).

- If arbitrary convex objects have disjoint bounding boxes, then $g_d(S)$ is at most $2^{d-1}$, as opposed to $\Omega(n^{d-1})$. 
Congruent Balls: Diameter

Lemma: Largest distance between pairs of ball centers is $\Omega(n)$.

- Let $D = \text{dist}(o_1, o_2)$, where $o_1, o_2$ centers with max distance.

- If $\ell$ a line transversal, then projection of $S$ onto $\ell$ has length at most $2 + D$.

- $S$ contained in cylinder of height $2 + D$ and radius 2.

- If $V_d$ volume of $d$-dim unit ball, then $(2 + D)2^{d-1}V_{d-1} \geq nV_d$.

- $D \geq \frac{nV_d}{2^{d-1}V_{d-1}} - 2 = \Omega(n)$.  

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Congruent Balls: Angles

Lemma: $\sin \theta \leq 2/D$, where $\theta$ the angle between a line transversal of $B_1, B_2$, and the line joining their centers; $D$ the distance between centers.

Lemma: If $z$ axis the line joining farthest pairs of centers in $S$, and $\ell$ a line transversal, then the angle between $z$ and $\ell$ is $O(1/n)$. 
Congruent Balls: Switched Pairs

- $(B_1, B_2)$ called a switched pair if there are two transversals $\ell$ and $\ell'$ that meet $B_1, B_2$ in different orders.

Lemma: The distance between a switched pair of balls is $O(1/n^2)$.

Lemma: If $(B_1, B_2)$ is a switched pair, and $\ell$ is a line transversal, then angle between $\ell$ and $\overrightarrow{o_1o_2}$ is $\pi/2 - O(1/n)$. 
Distance Lemma

Lemma: \((B_1, B_2)\) a switched pair. One line transversal \((z, 0, 0, \ldots, 0)\), and another \((z, az + b, c_3, \ldots, c_d)\). If \(z_0\) is the \(z\)-coordinate of the center of gravity of \((B_1, B_2)\), then

\[
|az_0 + b| \leq a.
\]

- \(a\) is the slope, and so \(a \approx \sin \theta = O(1/n)\).
- Transversal lines pass very close to the center of gravity of a switched pair.
Congruent Balls: Switched Pairs

Lemma: A ball appears in at most one switched pair.

- Assume two pairs \((B, B_1), (B, B_2)\), sharing a ball; \(g_i\) the midpoint of \(oo_i\); \(z\) the line thru farthest pair of centers.

- Let \(\pi\) be the plane through \(o\), \(\perp\) to \(\vec{z}\), then \(o_i, g_i\) lie close to \(\pi\).

- Any transversal passes close to \(g_i\).

- If \(\ell\) passes close to both \(g_1, g_2\), it is almost \(\parallel g_1g_2\), and so almost \(\perp \vec{z}\).

- Contradiction! since the angle between \(\ell\) and \(\vec{z}\) is \(O(1/n)\).
Implications

Corollary: If $k$ switched pairs, then $2^k$ permutations.

Lemma: Given two switched pairs, their centers of gravity have distance at least $\sqrt{2} - \epsilon(n)$. 
Upper Bound

Theorem: A set of congruent balls $S$ in $\mathbb{R}^d$ admits at most 2 switched pairs, where $|S|$ is sufficiently large depending on $d$.

- Transversals $(z, 0, \ldots, 0), (z, az + b, c_3, \cdots, c_d)$.
- If $z_1$ and $z_m$ centers of gravity, then $|az_1 + b| \leq a$ and $|az_m + b| \leq a$.
- $|z_1 - z_m| \leq 2$.
- If $m$ pairs, then $m - 1 \leq \frac{2}{\sqrt{2}} + \epsilon(n) \leq 1$. 
Rectangular Boxes

- $S$ a set of $n$ disjoint, axis-parallel rectangular boxes in $\mathcal{R}^d$.

- Two disjoint boxes $P, Q$ always separable by a hyperplane normal to some coordinate axis.

- $\text{sign}(x) = 1$ if $x \geq 0$, and $-1$ otherwise.

- $\text{sign}(x) = (\text{sign}(x_1), \ldots, \text{sign}(x_d))$.

- For a line transversal $\ell$, let $\vec{\ell}$ be its orientation vector.

Lemma: If $\text{sign}(\vec{\ell}) = \text{sign}(\vec{\ell'})$, then $\ell$ and $\ell'$ induce the same linear ordering on $S$. 
Theorem: \( n \) disjoint axis-parallel boxes admit at most \( 2^{d-1} \) permutations.

- If \( \ell \) a transversal, then \( \text{sign}(\ell) \in \{1,-1\}^d \).
- Exactly \( 2^d \) elements in \( \{1,-1\}^d \), each corresponding to at most one linear ordering of \( S \).
- Each geometric permutation correspond to two linear orderings, and lemma follows.
- Matching lower bound, even for cubes.
Extensions

Corollary: If $n$ convex objects in $\mathbb{R}^d$ are such that their smallest enclosing bounding boxes are pairwise disjoint, then $g_d(n) = 2^{d-1}$.

- General convex objects can have $\Omega(n^{d-1})$ permutations. Corollary shows effect of well-separation.

Lemma: If $S$ has a separating set of size $h$, then $S$ admits at most $2^{h-1}$ permutations.

- Using separating hyperplanes, Wenger [W90] gives a bound of $O(h^{d-1})$. 
Open Problems and New Results

- Congruent balls: what’s the right answer?

- Disjoint bounding boxes give $O(1)$ permutations, as opposed to $\Omega(n^{d-1})$ for convex bodies. Other natural conditions?

- Unit balls have $O(1)$ permutations, while general balls have $\Omega(n^{d-1})$. Dependence on ratio of radii?

- If the ratio between the largest and smallest balls of $S$ is $\gamma$, then the number of permutations is $O(\gamma^{\log \gamma})$.

- Improve the bound.