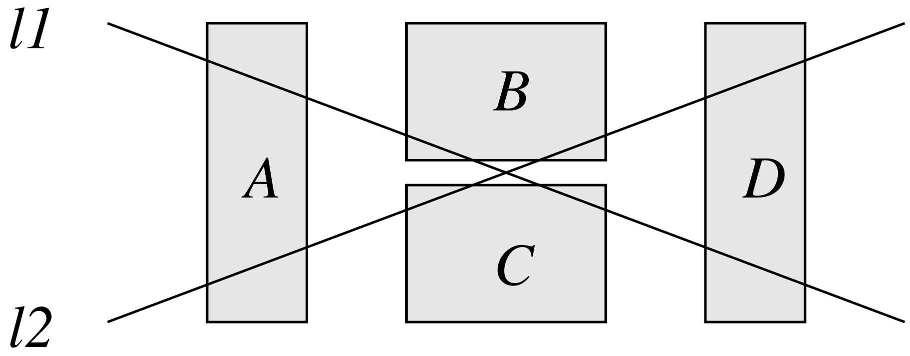
Shape Sensitive Geometric Permutations

Subhash Suri

Computer Science Department
UC Santa Barbara

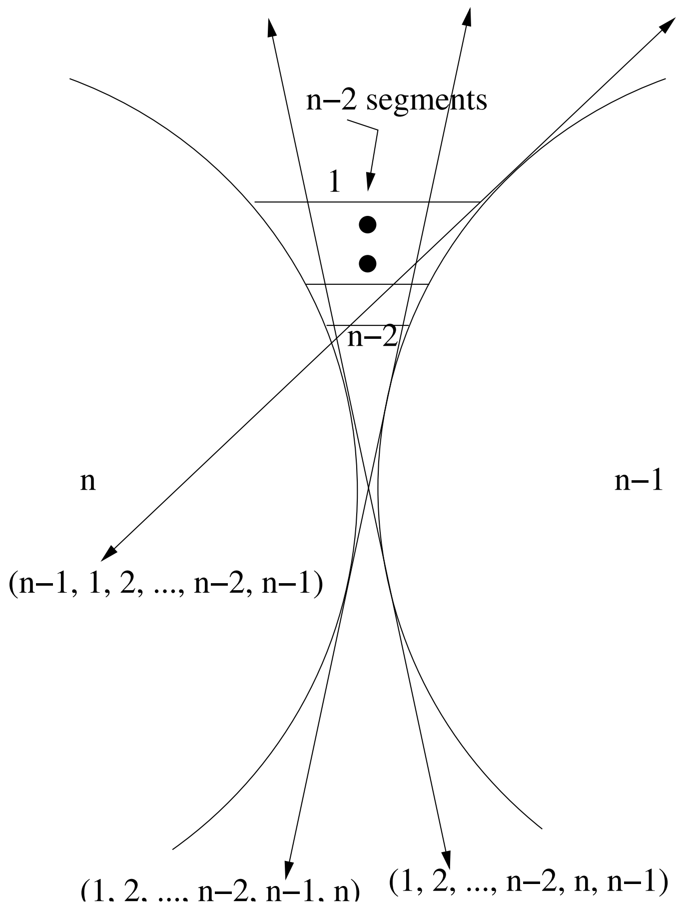
Joint work with Yunhong Zhou.

Geometric Permutations



- **Disjoint convex objects $\mathcal{S}$ in $\mathcal{R}^d$.**
- **ℓ a *line transversal* if it intersects all objects of $\mathcal{S}$.**
- **The order induced by ℓ is a *geometric permutation.* (Line is undirected.)**
- **Example shows two permutations: (A, B, C, D) and (A, C, B, D).**
- **How many permutations?**

Two Dimensions



- **n convex objects in 2D with $2n-2$ distinct permutations. [KLZ 85]**
- **$2n-2$ also an upper bound. [ES 90]**

Known Bounds

- $g_d(\mathcal{S})$ **is number of permutations for** $\mathcal{S}$**.**
- $g_d(n) \;=\; \max\{g_d(\mathcal{S}) \;\mid\; |\mathcal{S}| = n\}$**.**
- **The following bounds are known:**

  1. $g_d(n) \;=\; \Omega(n^{d-1})$ **[KLL 92].**
  2. $g_d(n) \;=\; O(n^{2d-2})$ **[W 90].**

- **Large gap between upper and lower bounds.**
- **Specialized family of convex objects, *balls*, [SMS 00].**

  1. $g_d(n) \;=\; \Theta(n^{d-1})$ **for balls in** $\mathcal{R}^d$**.**
  2. **For** $d = 2$**, *congruent* disks *same radius* have at most 2 permutations.**

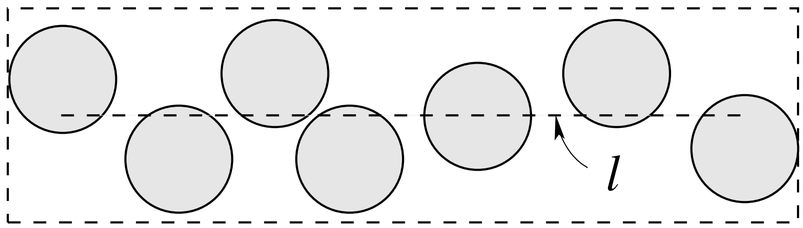
- **Conjecture:** $g_d(n) = O(1)$ **for congruent balls.**

New Results: Shape Sensitivity

- **A set of n unit balls in $\mathcal{R}^d$ admits at most *4 permutations*, with n depending on d.**
- **A set of n arbitrary size, but axis-aligned, rectangular boxes in $\mathcal{R}^d$ admits at most 2^{d-1} permutations.**
- **A matching lower bound of 2^{d-1} for boxes (cubes).**
- **If arbitrary convex objects have disjoint bounding boxes, then $g_d(\mathcal{S})$ is at most 2^{d-1}, as opposed to $\Omega(n^{d-1})$.**

Congruent Balls: Diameter

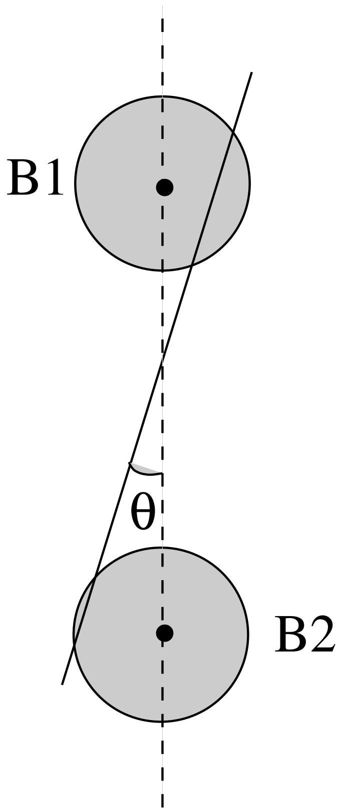
Lemma: ***Largest distance between pairs of ball centers is*** $\Omega(n)$***.***



- **Let** $D = \mathrm{dist}(o_1, o_2)$**, where** o_1, o_2 **centers with max distance.**
- **If** ℓ **a line transversal, then projection of** $\mathcal{S}$ **onto** ℓ **has length at most** $2 + D$**.**
- $\mathcal{S}$ **contained in cylinder of height** $2 + D$ **and radius 2.**
- **If** V_d **volume of** d**-dim unit ball, then** $(2 + D)2^{d-1}V_{d-1} \;\geq\; nV_d.$
- $D \;\geq\; \frac{nV_d}{2^{d-1}V_{d-1}} \;-\; 2 \;=\; \Omega(n)$**.**

Congruent Balls: Angles

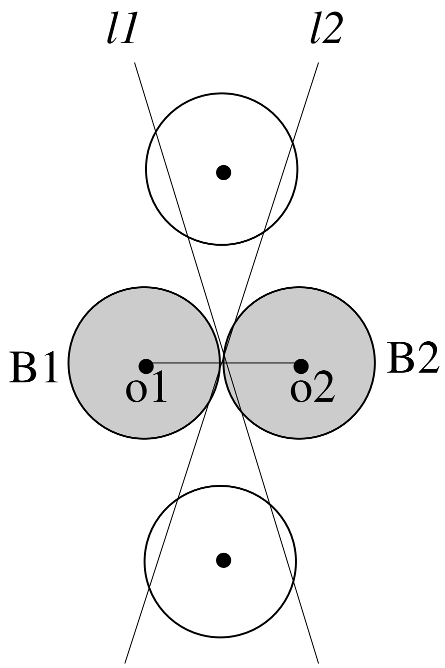
Lemma: $\sin\theta \leq 2/D$, ***where θ the angle between a line transversal of B_1, B_2, and the line joining their centers; D the distance between centers.***



Lemma: ***If z axis the line joining farthest pairs of centers in $\mathcal{S}$, and ℓ a line transversal, then the angle between z and ℓ is $O(1/n)$.***

Congruent Balls: Switched Pairs

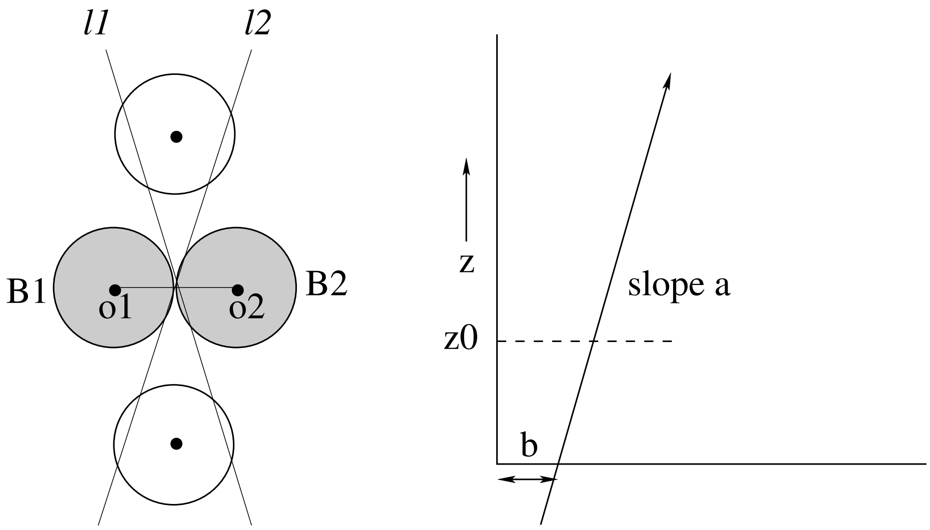
- (B_1, B_2) **called a *switched pair* if there are two transversals ℓ and ℓ' that meet B_1, B_2 in different orders.**



Lemma: ***The distance between a switched pair of balls is $O(1/n^2)$.***

Lemma: ***If (B_1, B_2) is a switched pair, and ℓ is a line transversal, then angle between ℓ and $\overrightarrow{o_1 o_2}$ is $\pi/2 - O(1/n)$.***

Distance Lemma



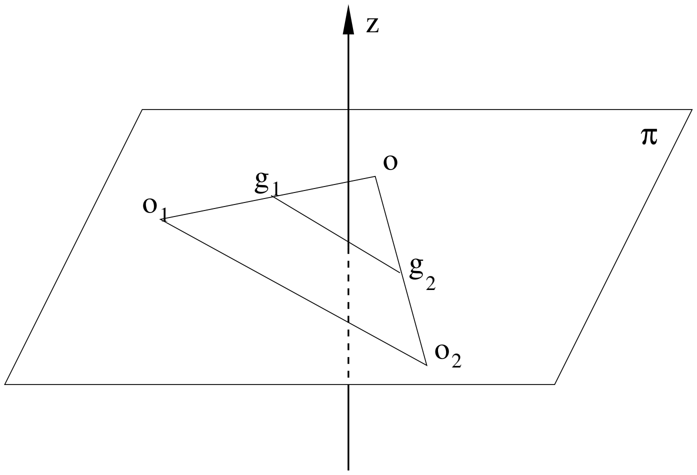
Lemma: (B_1, B_2) ***a switched pair. One line transversal*** $(z, 0, 0, \ldots, 0)$***, and another*** $(z, az + b, c_3, \cdots, c_d)$***. If*** z_0 ***is the*** z***-coordinate of the center of gravity of*** (B_1, B_2)***, then***

$$|az_0 + b| \leq a.$$

- a **is the slope, and so** $a \approx \sin\theta = O(1/n)$**.**
- **Transversal lines pass very close to the center of gravity of a switched pair.**

Congruent Balls: Switched Pairs

Lemma: ***A ball appears in at most one switched pair.***



- **Assume two pairs (B, B_1), (B, B_2), sharing a ball; g_i the midpoint of oo_i; z the line thru farthest pair of centers.**

- **Let π be the plane through o, $\perp$ to $\vec{z}$, then o_i, g_i lie close to π.**

- **Any transversal passes close to g_i.**

- **If ℓ passes close to both g_1, g_2, it is almost $\parallel$ g_1g_2, and so almost $\perp$ $\vec{z}$.**

- **Contradiction! since the angle between ℓ and $\vec{z}$ is $O(1/n)$.**

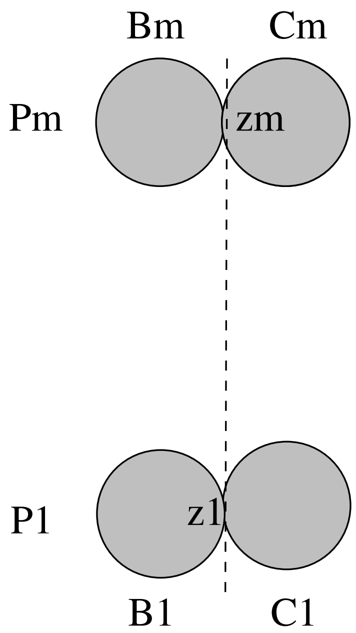
Implications

Corollary: ***If k switched pairs, then 2^k permutations.***

Lemma: ***Given two switched pairs, their centers of gravity have distance at least $\sqrt{2} - \epsilon(n)$.***

Upper Bound

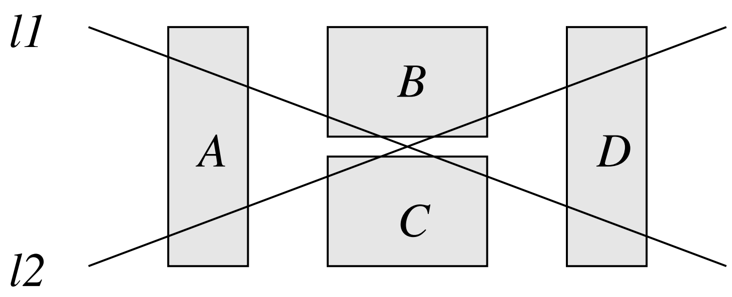
Theorem: ***A set of congruent balls $\mathcal{S}$ in $\mathcal{R}^d$ admits at most 2 switched pairs, where $|\mathcal{S}|$ is sufficiently large depending on d.***



- **Transversals** $(z, 0, \ldots, 0)$, $(z, az + b, c_3, \cdots, c_d)$.
- **If** z_1 **and** z_m **centers of gravity, then** $|az_1 + b| \leq a$ **and** $|az_m + b| \leq a$.
- $|z_1 - z_m| \leq 2$.
- **If** m **pairs, then** $m - 1 \leq \frac{2}{\sqrt{2}} + \epsilon(n) \leq 1$.

Rectangular Boxes

- **$\mathcal{S}$ a set of n disjoint, axis-parallel rectangular boxes in $\mathcal{R}^d$.**



- **Two disjoint boxes P, Q always separable by a hyperplane normal to some coordinate axis.**

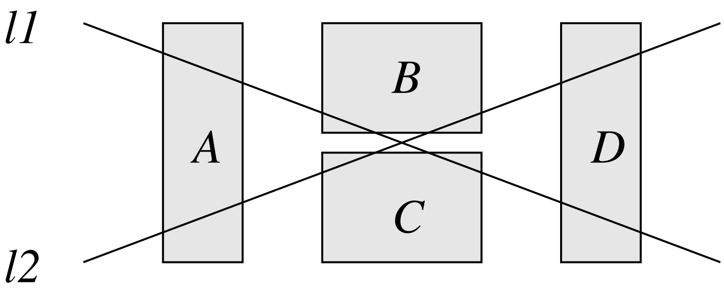
- **sign**(x) = **1 if $x \geq 0$, and -1 otherwise.**

- **sign**(x) = (**sign**$(x_1), \cdots,$ **sign**(x_d)).

- **For a line transversal ℓ, let $\vec{\ell}$ be its orientation vector.**

Lemma: ***If sign$(\vec{\ell})$ = sign$(\vec{\ell'})$, then ℓ and ℓ' induce the same linear ordering on $\mathcal{S}$.***

Rectangular Boxes



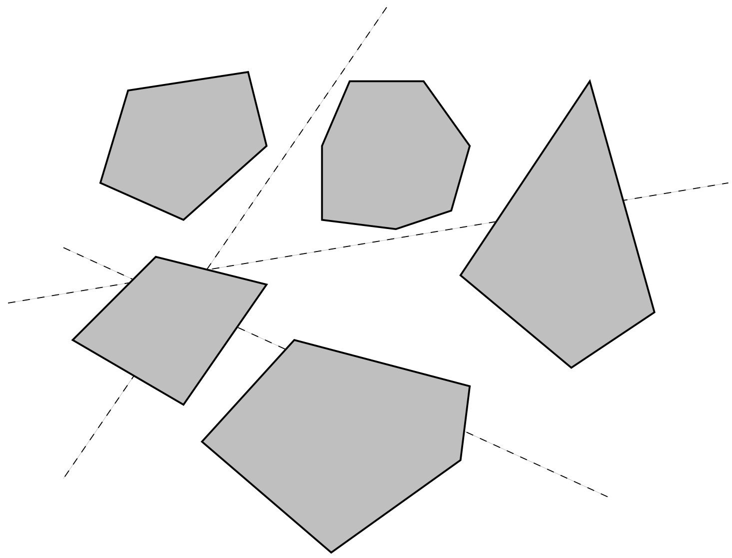
Theorem: n ***disjoint axis-parallel boxes admit at most*** 2^{d-1} ***permutations.***

- **If** ℓ **a transversal, then sign**$(\vec{\ell}) \in \{1, -1\}^d$**.**
- **Exactly** 2^d **elements in** $\{1, -1\}^d$**, each corresponding to at most one linear ordering of** $\mathcal{S}$**.**
- **Each geometric permutation correspond to two linear orderings, and lemma follows.**
- **Matching lower bound, even for cubes.**

Extensions

Corollary: ***If n convex objects in $\mathcal{R}^d$ are such that their smallest enclosing bounding boxes are pairwise disjoint, then $g_d(n) = 2^{d-1}$.***

- **General convex objects can have $\Omega(n^{d-1})$ permutations. Corollary shows effect of well-separation.**



Lemma: ***If $\mathcal{S}$ has a separating set of size h, then $\mathcal{S}$ admits at most 2^{h-1} permutations.***

- **Using separating hyperplanes, Wenger [W90] gives a bound of $O(h^{d-1})$.**

Open Problems and New Results

- **Congruent balls: what's the right answer?**
- **Disjoint bounding boxes give $O(1)$ permutations, as opposed to $\Omega(n^{d-1})$ for convex bodies. Other natural conditions?**
- **Unit balls have $O(1)$ permutations, while general balls have $\Omega(n^{d-1})$. Dependence on ratio of radii?**
- **If the ratio between the largest and smallest balls of $\mathcal{S}$ is γ, then the number of permutations is $O(\gamma^{\log \gamma})$.**
- **Improve the bound.**