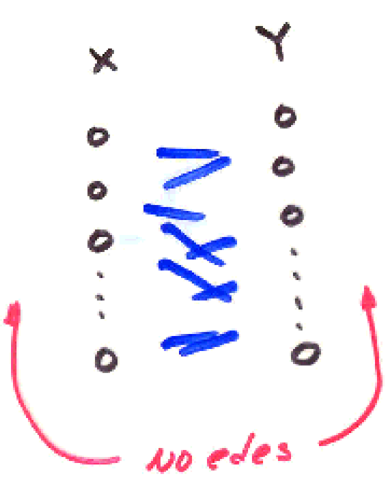
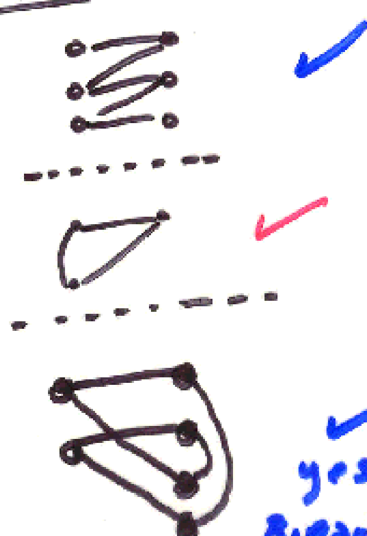
Slides.NP-Bipartite.Matching, Teofilo F. Gonzalez UCSB

GRAPH MATCHING

* GRAPH $G=(V,E)$ IS BIPARTITE IF THE SET OF VERTICES V CAN BE PARTITIONED INTO TWO SETS $(X + Y)$ SUCH THAT NO TWO VERTICES IN THE SAME SET ARE ADJACENT.

EXAMPLE



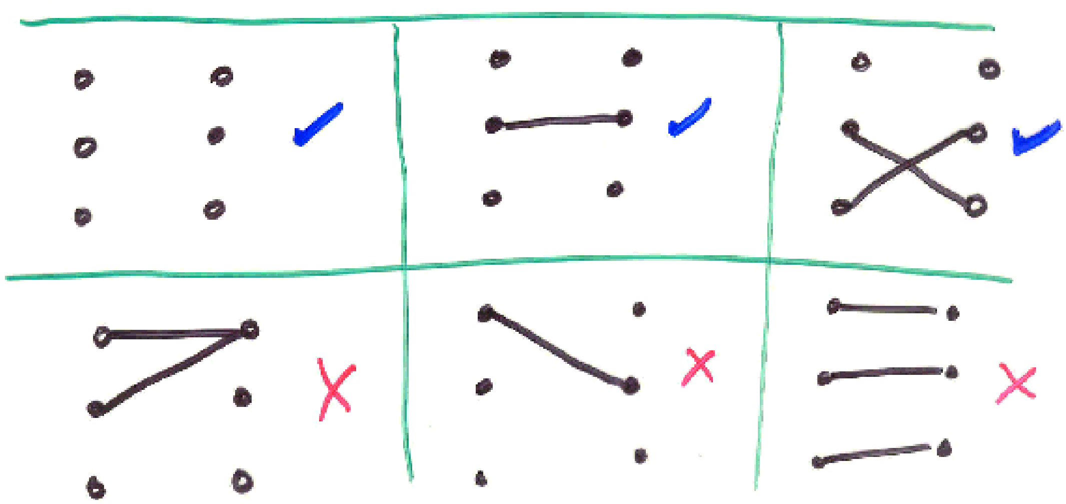


yes!! Bipartite

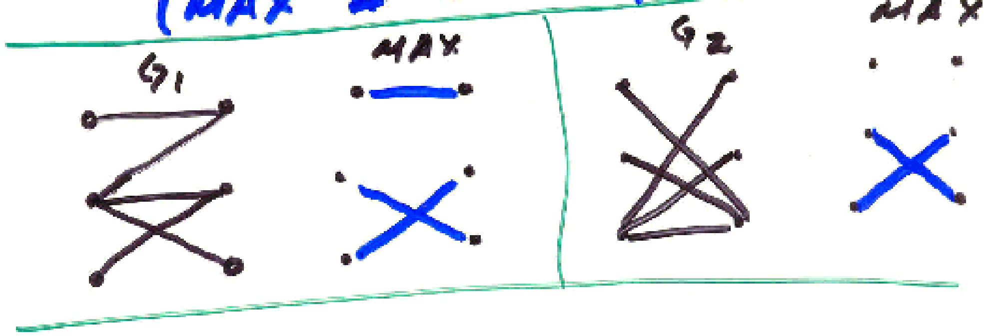
MATCHING: SUBSET OF EDGES OF E, NO TWO OF WHICH ARE INCIDENT UPON THE SAME VERTEX.

G

MATCHINGS



MAXIMUM MATCHING: Matching of MAXIMUM CARDINALITY (MAX # of edges)



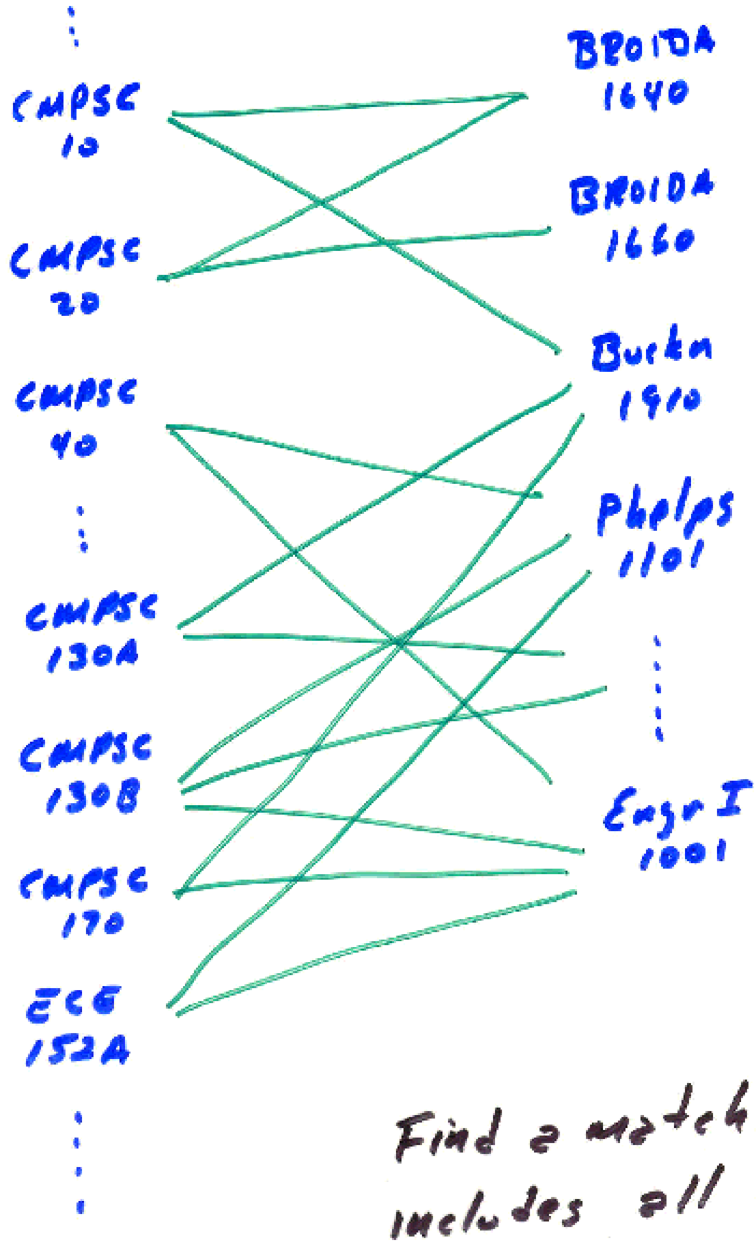
COMPLETE MATCHING: EVERY VERTEX IS AN ENDPOINT OF SOME EDGE IN THE MATCHING.

G_1 has a complete matching

G_2 does not have a complete matching

Application

MWF 1-2



Find a matching that includes all courses.

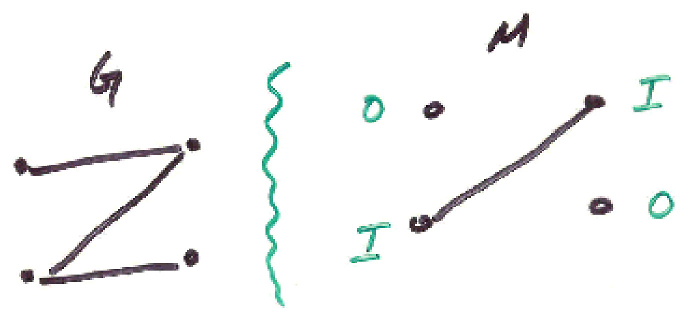
MAXIMUM Matching Algorithm

Algorithmic Technique (cmpsc 130C)

Augmentation: Start with a solution (maybe an empty one) and continue improving it until no further improvement is possible.

Augmenting path with respect to a matching M:

- A node is labeled I if there is an edge incident to it in matching M.
- A node is labeled O otherwise



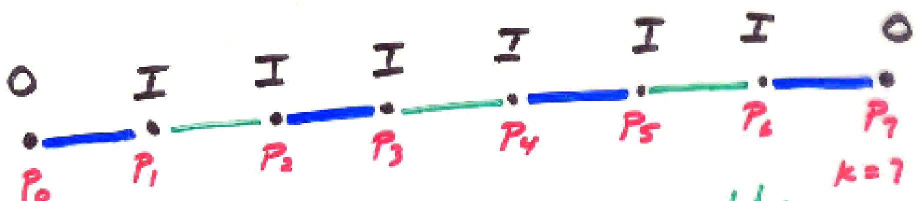
A simple path $P = P_0, P_1, P_2, \ldots P_k$, where $P_0, P_1, \ldots P_k$ are the vertices it visits, is said to be an augmenting path relative to M iff

(a) P_0 + P_k are labeled O

(b) $P_1, P_2, \ldots P_{k-1}$ are labeled I

(c) Edges $\{P_0, P_1\}, \{P_2, P_3\}, \{P_4, P_5\}, \ldots$ are in the graph but not in the matching

(d) Edges $\{P_1, P_2\}, \{P_3, P_4\}, \{P_5, P_6\}, \ldots$ are in the matching.

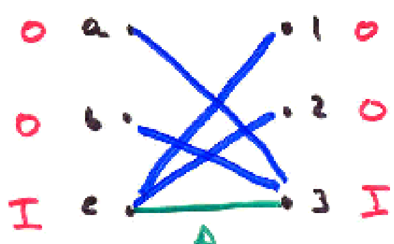


GREEN Edges are in matching

Blue Edges are in the graph but not in the matching

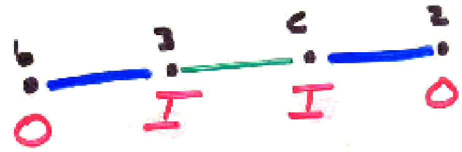
BG.8

G



matching is (just one edge)

Augmenting path P



Operation $M \oplus P \rightarrow M'$

"exclusive or"

"OR operation without elements in both sets"

$M = \{c, 3\}$

$P = \{\{b, 3\} \{3, c\} \{c, 2\}\}$

$M' = M \oplus P = \{\{b, 3\}, \{c, 2\}\}$

Maximum Matching

Algorithm (high level)

$M \leftarrow \emptyset$

while there exists an augmenting path for M do

Let P be an augmenting path relative to M

$M \leftarrow M \oplus P$

endwhile

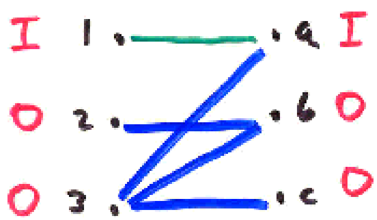
end algorithm

BG. 10

$M_0 = \emptyset$

$P = \{1, a\}$

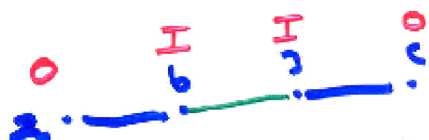
$M_1 = M_0 \oplus P = \{1, a\}$

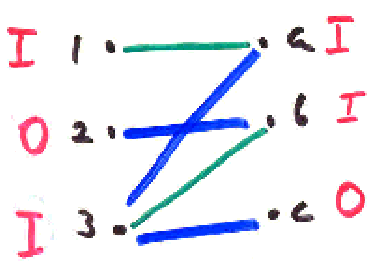


$P = \{3, b\}$

$M_2 = M_1 \oplus P$

$= \{\{1,a\}, \{3,c\}\}$

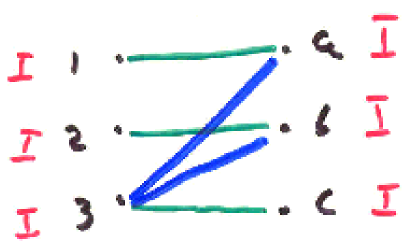




$P = \{\{2,b\}, \{b,3\}, \{3,c\}\}$

$M_3 = M_2 \oplus P$

$= \{\{1,a\}, \{2,b\}, \{3,c\}\}$



Correctness

Theorem: If M is not a maximum matching, then there is an augmenting path relative to M.

Proof: Bipartite graph is $G=(V,E)$

Let M be a Matching that is not maximum

Let N be a maximum matching

clearly, $|N| > |M|$

Let $G' = (V, M \oplus N)$

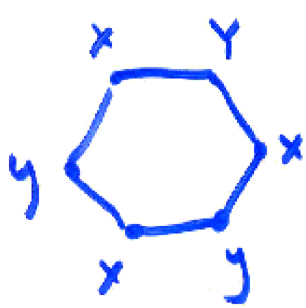
clearly, $|M \oplus N| > 0$

Since every edge in G' is
part of matching M or
part of matching N,
EVERY NODE in G' is
of degree 0, 1, or 2.

So the components of G'
are

single nodes

simple paths

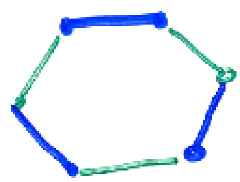


cycles with
even # of
edges

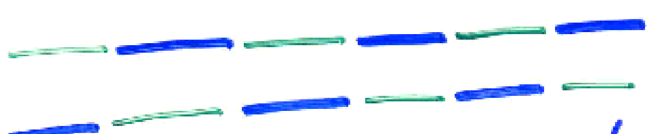
BG.13

In G' color green the edges from M and blue the edges from N.

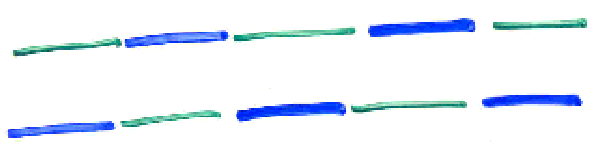
Every cycle in G' has the same # of green + blue edges



Every path with an even # of edges has same # of green + blue edges



Paths with odd # of edges



$\#g > \#b$

$\#b > \#g$

* Initially in G'

# of blue edges > # of green edges
& there is at least one edge

* delete from G' all cycles & paths
with an even # of edges

new G' has

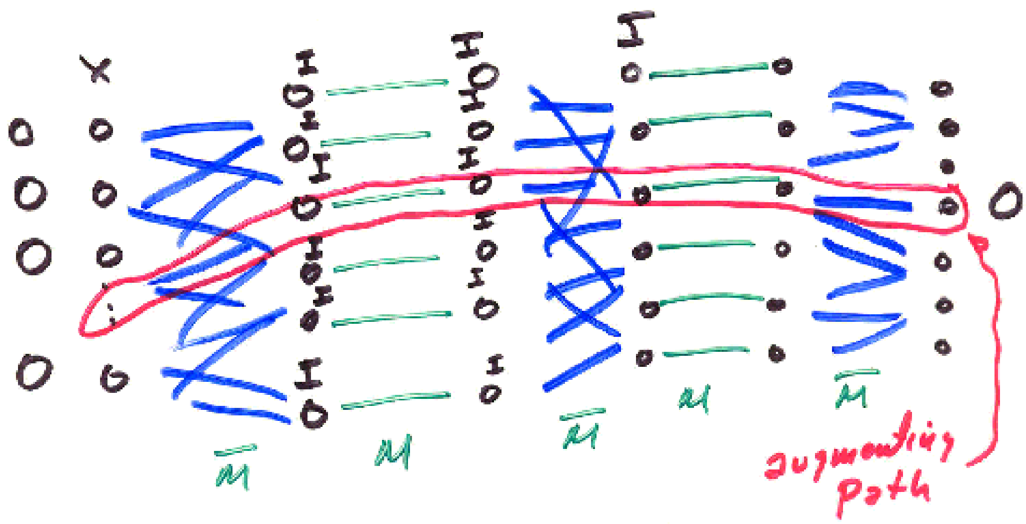
# of blue edges > # of green edges
& there is at least one edge

* delete from G' all paths with
# of edges & with
an odd # of blue edges >
# of green edges >

new G' has
# of blue edges > # of green edges
& there is at least one edge

⇒ there is an augmenting path □

construction of augmenting path BG.15



finding augmenting path $O(e)$, e # of edges in graph

algorithm finds at most n augmenting paths ($2n$ nodes in G)

Time complexity $\Rightarrow$ is $O(ne)$.

* there are faster methods $O(e\sqrt{n})$