Self-Adjusting Heaps

- No explicit structure. Adjust the structure in a simple, uniform way, so that the efficiency of future operations is improved.

Amortized Time Complexity

- Total time for operations / number of operations.

Example: Amortized Complexity

Let `S` be an array (with $n+1$ elements) and `top` be an nonnegative integer. We will use `S` and `top` to represent a stack. Initially `top = 0`. There are three operations on the stack: `Push, Pop and Multipop`. These operations are defined as follows:

```
Push (x)
  top++;
  S[top] = x;
 end Push;

 Pop (x)
  if (top == 0) then return;
  print S[top];
  top--;
  return;
 end Pop;
```

```
Multipop (k)
  for i = 1 to k do
   {if (top == 0) then return;
   print S[top];
   top--;}
  return
 end Multipop;
```

What is the worst case time complexity for `Push(x)`, `Pop(x)`, and `Multipop(k)`?

Executing any sequence of n operations of the form `Push(x)`, `Pop(x)`, and `Multipop(k)` takes time equal to n times the worst time complexity of executing any of the above three operations.

Is the bound best possible (i.e., is it tight)?

Comparison

- **Worst Case TC:** Insert $O(x)$ and Delete $O(y)$: Every time the algorithm is run each Insert operation takes $O(x)$ and each Delete operation takes $O(y)$.
- **Average Case TC:** Insert $O(x)$ and Delete $O(y)$: When the algorithm is run over a set of inputs with a given frequency count the Insert operation takes on average $O(x)$ and the Delete operation takes on average $O(y)$.
- **Amortized TC:** Insert $O(x)$ and Delete $O(y)$: Every time the algorithm is run the Insert operation takes on average $O(x)$ and the Delete operation takes on average $O(y)$.

Mergeable Heap

- ADT defined over a totally ordered universe. Operations are:
- **Make heap**(h): Create a new, empty heap, named h.
- **Find Min**(h): Return the min item in heap h. If h is empty then return the special item called "null".
- **Insert**(x, h): Insert item x in heap h, not previously containing it.
- **Delete min**(h): Delete the minimum item from heap h, and return it. If the heap is initially empty then return "null".
- **Meld**(h_1, h_2): Return the heap formed by taking the union of disjoint heaps h_1 and h_2. This operation destroys h_1 and h_2.

Heap-Ordered Binary Tree (Skew Heaps)

Binary tree whose nodes are items.

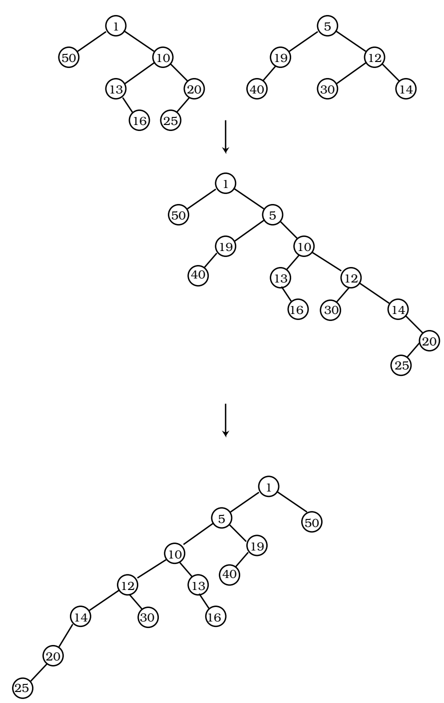
Tree is arranged in a heap order, if $p(x)$ is the parent of x, then the item stored at $p(x)$ is less than the item stored at x.

Implementation of Operations

- *Make Heap*: O(1) time by just setting the root of h to null.
- *Find Min*(h): Return the item stored in the root of h.
- *Insert*(x, h): Make x a single node heap and meld it with h.
- *Delete min*(h): Delete the root and replace h with the meld of its left and right

Meld(h_1, h_2)

- Form a single tree by traversing the right paths of h_1 and h_2, merging them into a single right path with items in increasing order.
- The left subtrees of nodes along the **merge path** do not change.
- Swap the left and right children of every node on the merge path except at the lowest level.



MELD Algorithm

Procedure meld(val h_1, h_2**)**
  if $h_2 = null$ **then return** h_1
    else return xmeld(h_1, h_2);
end

Procedure xmeld(val h_1, h_2**)**
// h_2 is not null //
  if $h_1 = null$ **then return** h_2;
  if $item(h_1) > item(h_2)$ **then** $h_1 \leftrightarrow h_2$;
  (lchild(h_1), rchild(h_1)) $\leftarrow$
    (xmeld(rchild(h_1),h_2), lchild(h_1));
  return h_1
End of Procedure

Definitions

- S: Collection of Skew Heaps.
- $\Phi(S)$: Potential of S.
- m operations with times $t_1, t_2, ..., t_m$.
- a_i amortized time for operation i.
- Φ_i: Potential after operation i.
- Φ_0: Initial potential.
- $\sum t_i = \sum(a_i - \Phi_i + \Phi_{i-1}) = \Phi_0 - \Phi_m + \sum a_i$
- Φ_0 is initially zero.
- Φ_i is non-negative.

Idea

- High Potential: Remaining operations may be expensive.
- Low Potential: Remaining operations are inexpensive.
- Amortized bound: $O(\log\ n)$ time per operation.

Definitions

- $wt(x)$: Number of descendants of x (incl. x).
- Non-root x is *heavy* if $wt(x) > wt(p(x))/2$.
- Non-root x is *light* otherwise.
- Node x is *right* if it is a right child.
- Node x is *left* if it is a left child.

Results

Lemma 1: Of the children of any node, at most one is heavy.

Lemma 2: On any path from node x down to a descendant y, there are at most $\lfloor log\ (wt(x)/wt(y)) \rfloor$ light nodes, not counting x. In particular, any path in an n-node tree contains at most $\lfloor log\ n \rfloor$ light nodes.

Proof: If there are k light nodes not including x along the path from x to y, then

$$wt(y) \le wt(x)/2^k \Rightarrow$$

$$k \le log\ (wt(x)/wt(y)).$$

Potential of a Skew Heap: Total number of right heavy nodes in it.

Definitions

- Let n_1 and n_2 be the number of nodes in h_1 and h_2, resp.
- Number of light nodes on the right path of h_1 (h_2) is at most $\lfloor log\ n_1 \rfloor$ ($\lfloor log\ n_2 \rfloor$).
- Let k_1 and k_2 be the number of heavy nodes on the right path of h_1 and h_2, resp.
- Let k_3 be the number of new right heavy nodes in the resulting heap. Clearly $k_3 \leq \lfloor log\ n \rfloor$

Bounds

- Number of nodes on the merge path is **at most** $2 + \lfloor log\ n_1 \rfloor + k_1 + \lfloor log\ n_2 \rfloor + k_2 \leq 1 + 2\lfloor log\ n \rfloor + k_1 + k_2$
- Increase in potential because of the meld is $k_3 - k_1 - k_2 \leq \lfloor log\ n \rfloor - k_1 - k_2$
- Amortized cost is $3\lfloor log\ n \rfloor + 1$.