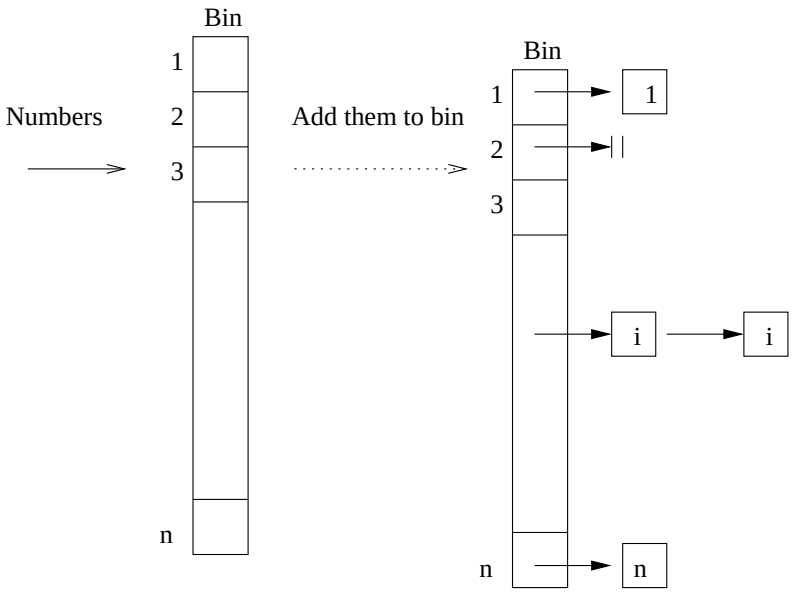
Bin Sort

- Sorting integers in Range $[1, \ldots, n]$
- Add all elements to table and then
- Retrieve in order $1, 2, 3, \ldots, n$
- "Stable" Sorting Method (repeated elements will end up in their original order)



Radix Sort

```
Input list
216 521 425 116 91 515 124 34 96 24

After sorting on least significant digit
521 91 124 34 24 425 515 216 116 96

After sorting on 2nd least significant digit
515 216 116 521 124 24 425 34 91 96

After sorting on 3rd least significant digit
24 34 91 96 116 124 216 425 515 521
```

- Sort n numbers whose values are in [0,1,...,n-1] can be done in $O(n)$ time via bin sort (use n bins).
- Sort n numbers whose values are in [0,1,..., $n^2 - 1$] can be done on $O(n)$ time via radix sort. Each key is $\log n$ bits long. There are two keys per value.
  $\underbrace{xxxx}_{key\#2}\underbrace{xxxx}_{key\#1}$ ← number in $[0, \ldots, n^2 - 1]$ have $2 \log n$ bits.
- Sort n numbers whose values are in [0,1,..., $n^k - 1$] can be done in $O(kn)$ time via radix sort. Each key is $\log n$ bits long. There are k keys per value.
  $\underbrace{xxxx}_{key\#k}\ \underbrace{xxxx}_{key\#k-1}\ \ldots\underbrace{xxxx}_{key\#1}$ ← number in $[0, \ldots, n^k - 1]$ have $k \log n$ bits.

Heap Sort

Sort $x_1, x_2, \ldots, x_n$

```
for i=1 to n
  insert x_i in Heap
for i=1 to n
  delete min from Heap
```

Time complexity is $O(n \log n)$.
As we discussed before, the first phase (insert n elements) can be done in $O(n)$ time. (bottom up)
Sorting method is Not Stable

Merge Sort

- [8][4][5][6][2][1][7][3] in array A
- $[4,8][5,6][1,2][3,7]$ in array B, TC is $O(n)$
- $[4,5,6,8][1,2,3,7]$ in array A, TC is $O(n)$
- $[1,2,3,4,5,6,7,8]$ in array B, TC is $O(n)$
- Number of passes is $\log_2 n$
- Time complexity is $T(n) = 2T(n/2) + cn$ when considering it top down. Solution to this recurrence relation is $O(n \log_2 n)$, and uses $O(n)$ additional space.
- Stable Sorting method

Merge Sort Analysis

Assume, $n = 2^k$ for some positive integer k

$$T(n) = \begin{cases} T(1) & n = 1 \\ 2T(n/2) + cn & n > 1 \end{cases}$$

$$\begin{aligned} T(n) &\leq 2T(n/2) + cn \\ &= 2(2T(n/4) + cn/2) + cn \\ &= 4T(n/4) + 2cn \\ &= 4(2T(n/8) + cn/4) + 2cn \\ &= 8T(n/8) + 3cn \ldots \text{ (Stop when } n/2^k = 1) \\ &= 2^k T(1) + kcn \\ &= n + cn \log n \\ &\quad O(n \log n) \end{aligned}$$

if $2^k < n \leq 2^{k+1}$, $T(n) \leq T(2^{k+1}) \leq T(2n)$, which is $O(n \log n)$

Quick Sort

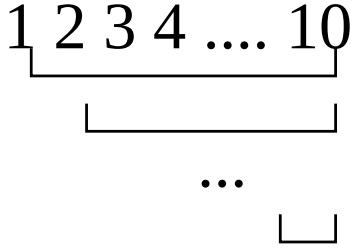
```
template<class T> T
void QuickSort(T a[],int l,int r)
{// Sort a[l:r], a[r+1] has large value.
    if (l >= r) return;
    int i = l,      // left to right cursor
        j = r + 1;  // right to left cursor
    T pivot = a[l]; // Partition wrt pivot
    while (true) {
       do {// find >= element on left side
          i = i + 1; } while (a[i] < pivot);
       do {// find <= element on right side
          j = j - 1; } while (a[j] > pivot);
       if (i >= j) break;//swap pair not found
       Swap(a[i], a[j]); }
    a[l] = a[j]; a[j] = pivot;
    QuickSort(a, l, j-1);//sort left segment
    QuickSort(a, j+1, r);//sort right segment
}
```

QuickSort

- A[r+1] has value larger than all other elements. Why?
- Time complexity bound is $O(n^2)$.
- $O(n)$ additional space
- NONSTABLE Sorting method

QuickSort

- Worst Case



  - $\sum_{i=1}^{n} i$ which is $O(n^2)$
  - Many instances take $\Omega(n^2)$
  - NONSTABLE Sorting Method

- Space Complexity: (QS(n) = QuickSort on n elements)

$$\overbrace{QS(n), QS(n-1), QS(n-2), \ldots, QS(1)}^{\Omega(n)}$$

Reduce Space Complexity

- Make procedure non-recursive
- Solve the smaller problem first
- for example

  sort(1,80)

  $\Downarrow$ size n

  sort (1,40), sort(42,80)

  $\Downarrow size \leq n/2$

  sort(42,62), sort(64,80)

  $\Downarrow size \leq n/4$

  sort(64,74), sort(76,80)

  ...

  $\rightarrow \log_2(n)$ space

Quick Sort

Note: Changing the recursive program so that it calls the smallest subproblem first has space complexity Q(n).

```
QS(a[],l,r)
  ...
  if j-1-l <= r-(j+1)
    then QS(a,l,j-1)
         QS(a,j+1,r)
    else QS(a,j+1,r)
         QS(a,l,j-1)
end QS
```

Space n

$$\overbrace{QS(1,n)QS(2,n),QS(3,n),\ldots,QS(n-1,n)}^{\text{Recursive cells stack}}$$

Iterative Quick Sort

```
QS(...,1,n)
  S <- NULL
  PUSH (S,(1,n))
  while S is not Empty do
    (l,r) <- POP(S)
    if l >= r go to end while
    ...
    if j-1-l <= r-(j+1)
       then Push(S,(j+1,r)) //large
            Push(S,(l,j-1)) //small
       else Push(S,(l,j-1)) //large
            Push(S,(j+1,r)) //small
  endwhile
end QS
```

Iterative Quick Sort

Stack has length $O(\log n)$

Stack contents at different times for a problem with α values.

- α
- $< \alpha, \leq \alpha/2$
- $< \alpha, < \alpha/2, \leq \alpha/4$
- $< \alpha, < \alpha/2, < \alpha/4, \leq \alpha/8$
- ...

Stack has length $O(\log n)$.

Time Complexity

Worst Case

$$\begin{aligned} T(n) &\leq T(n-1) + cn \\ &= T(n-2) + c(n-1) + cn \\ &= \ldots \\ &= T(1) + (c)2 + (c)3 + \ldots + c(n-1) + c(n) \\ &= O(n^2) \end{aligned}$$

There are examples that show that $T(n)$ is $\Omega(n^2)$.

Best Case

T(n)=2T(n/2)+cn

$\rightarrow O(n \log n)$, like merge sort but constant is smaller

Average Case

$$
\begin{aligned}
T(n) &\leq T(i) + T(n-i-1) + cn \\
A(n) &= \frac{2}{n}\sum_{j=0}^{n-1}(A(j)) + cn \\
nA(n) &= 2\sum_{j=0}^{n-1} A(j) + cn^2
\end{aligned}
$$

Subtracting from last equation
$(n-1)A(n-1) = 2\sum_{j=0}^{n-2} A(j) + c(n-1)^2$
we know that

$$
\begin{aligned}
nA(n) - (n-1)A(n-1) &= 2A(n-1) + 2cn - c^{\dagger} \\
nA(n) &= (n+1)A(n-1) + 2cn \\
\frac{A(n)}{n+1} &= \frac{A(n-1)}{n} + \frac{2c}{n+1}
\end{aligned}
$$

†Note that $-c$ is not significant!

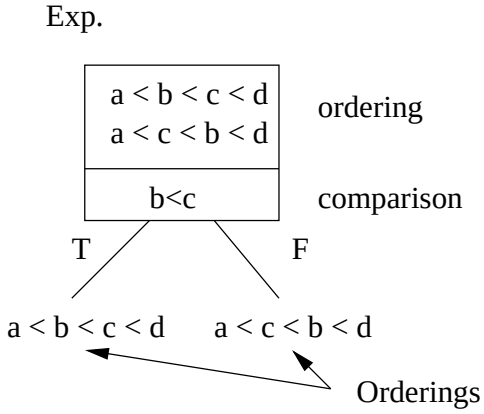
Adding the following equations

$$\begin{aligned}
\frac{A(n)}{n+1} &= \frac{A(n-1)}{n} + \frac{2c}{n+1} \\
\frac{A(n-1)}{n} &= \frac{A(n-2)}{n-1} + \frac{2c}{n} \\
\frac{A(n-2)}{n-1} &= \frac{A(n-3)}{n-2} + \frac{2c}{n-1} \\
&\vdots \quad \vdots \quad \vdots \\
\frac{A(2)}{3} &= \frac{A(1)}{2} + \frac{2c}{3}
\end{aligned}$$

We know that $\frac{A(n)}{n+1} = \frac{A(1)}{2} + 2c \sum_{j=3}^{n+1} \frac{1}{j}$
The last summation is $\log_e(n+1) + 0.577 - \frac{3}{2}$
Therefore, $\frac{A(n)}{n+1} = O(\log n)$
Which is equivalent to $A(n) = O(n \log n)$
Randomized (random pivot) has expected time complexity equal to $A(n)$.

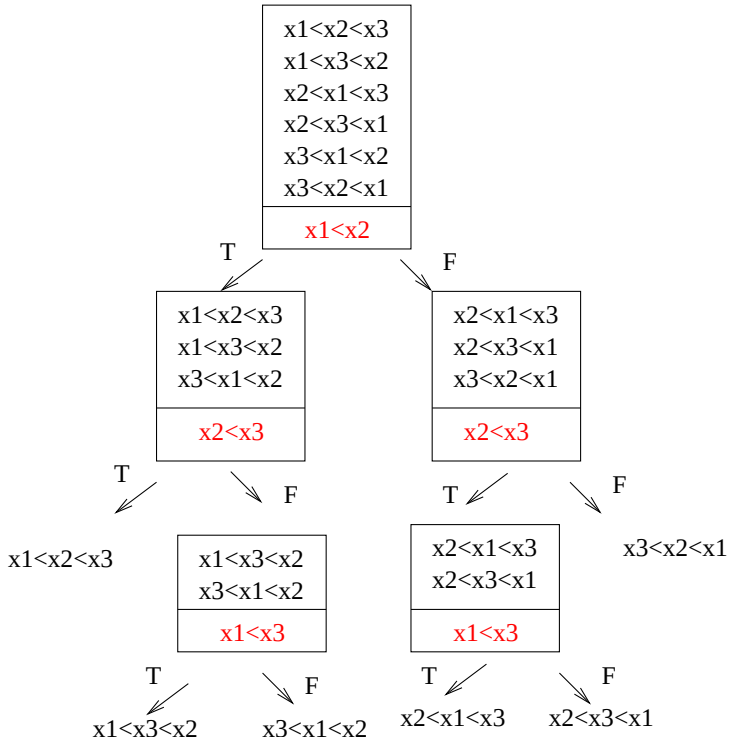
Lower Bounds for Sorting

- Decision Trees: Binary tree
  - Nodes represent a set of possible orderings
  - A comparison is made between elements
  - The result is two set of possible orderings



Assume elements are distinct

Every sorting method (that gains info only by comparing elements) may be viewed as a decision tree.



Lower Bound

- Sorting n elements $x_1, x_2, x_3, \ldots, x_n$
- $n!$ leaves in any decision tree
- Height h of decision tree
  $h \geq \log_2 n!$
  Since $n! \geq \left(\frac{n}{2}\right)^{n/2}$
  we know that h is $\Omega(n \log n)$

Sorting method that gain info only by comparing keys take time $\Omega(n \log n)$

Lower Bound for Average TC

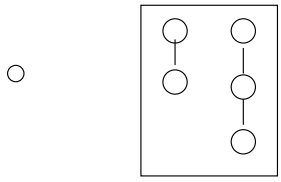
The average number of comparison (for decision tree sorting) is equal to the Average Depth of a Leaf

Theorem: For any decision tree with k leaves the $ADL \geq \log_2 k$.

Proof: Proof by Contradiction

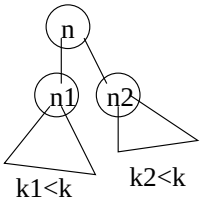
Let T be the least height decision tree with $ADL < \log_2 k$, where k=# of leaves in T.

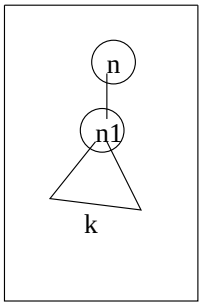
T: k = 1



Not least height

T: k >= 2





Not least height

Lower Bound for Average TC: Cont'

- Let $k = k_1 + k_2$, then by Ind. Hypothesis $ADL(n_1) \geq \log_2 k_1$, $ADL(n_2) \geq \log_2 k_2$, $k = k_1 + k_2$
- $\rightarrow ADL \geq \frac{k_1}{k_1+k_2} \log_2 k_1 + \frac{k_2}{k_1+k_2} \log_2 k_2 + 1$
- $\rightarrow$ min when $k_1 = k_2 = k/2$
- ADL is $\geq 0.5 \log k/2 + 0.5 \log k/2 + 1 = \log_2 k$
- A contradiction.
- Note that $(0.5+\epsilon)\log(k/2+\epsilon) + (0.5-\epsilon)\log(k/2-\epsilon) + 1$ is greater simply because
- $\log(k/2+\epsilon) - \log k/2 > \log k/2 - \log(k/2-\epsilon)$

Summary

Algorithm	WC	AC	Stable?	Space
Binsort	$O(n)$			$[0, n-1]$
RadixSort	$O(kn)$			$[0, n^k - 1]$
HeapSort	$O(n \log n)$	$0(n \log n)$	not stable	$O(c)$
QuickSort	$O(n^2)$	$O(n \log n)$	not stable	$O(\log n)$
Mergesort	$O(n \log n)$	$O(n \log n)$	stable	$O(n)$
Insertsort*	$O(n^2)$	$O(n^2)$	stable	

Insertsort*: Best when sorting ≤ 15 elements, used in hybrid algorithms

Sorting of Integers

- $O(n \log\log n)$ time: S. Nilsson, The fastest sorting algorithm, Dr. Dobbs Jr. Apr. 2000. Lots of extra space
- $O(n \log\log n)$ time and $O(n)$ space, "Not practical"
  Y. Han, Determinist Sorting in $O(n \log\log n)$ time + Linear Space, STOC'02, pp. 602-608.