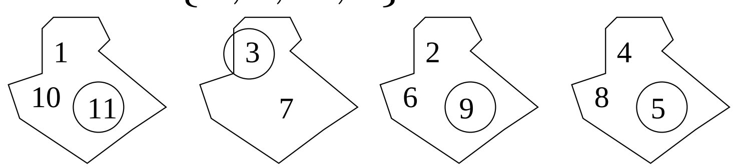
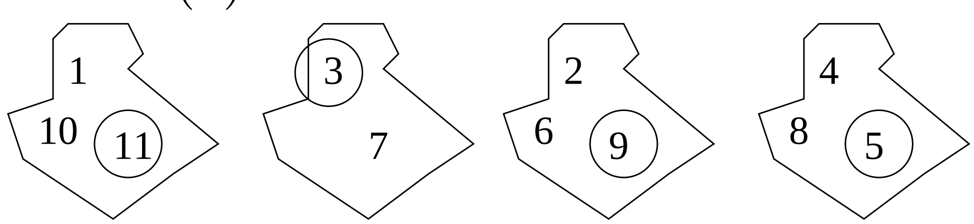
Disjoint Set Union

- HUGE number of other applications.
- Union(merge): Union of two sets
- Find: Find set where elements belongs
- Universe U={1,2,...,n}



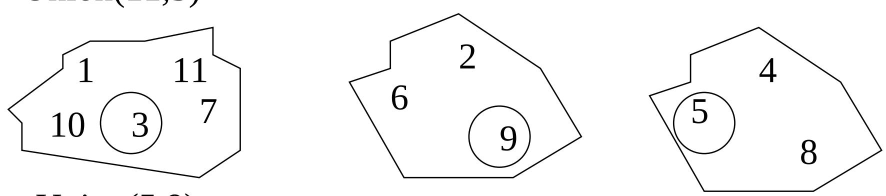
Circled number is name of set

- Find(j): find set to which j belongs.
  - Find(8) returns 5
  - Find(5) returns 5
  - Find(2) returns 9
  - Find(4) returns 5

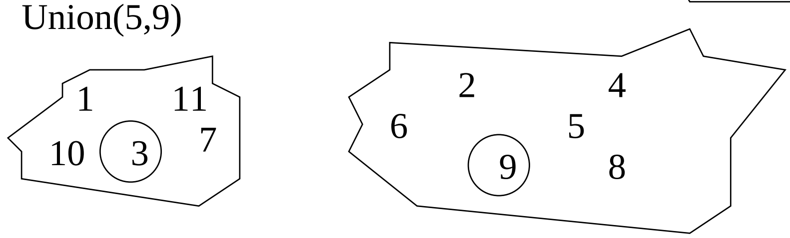


Circled number is name of set

Union(11,3)

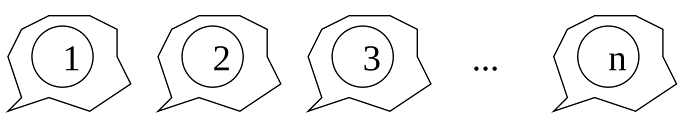


Union(5,9)



Note: Name of resulting set could be 5 or 9

Initial conditions:



Array Implementation

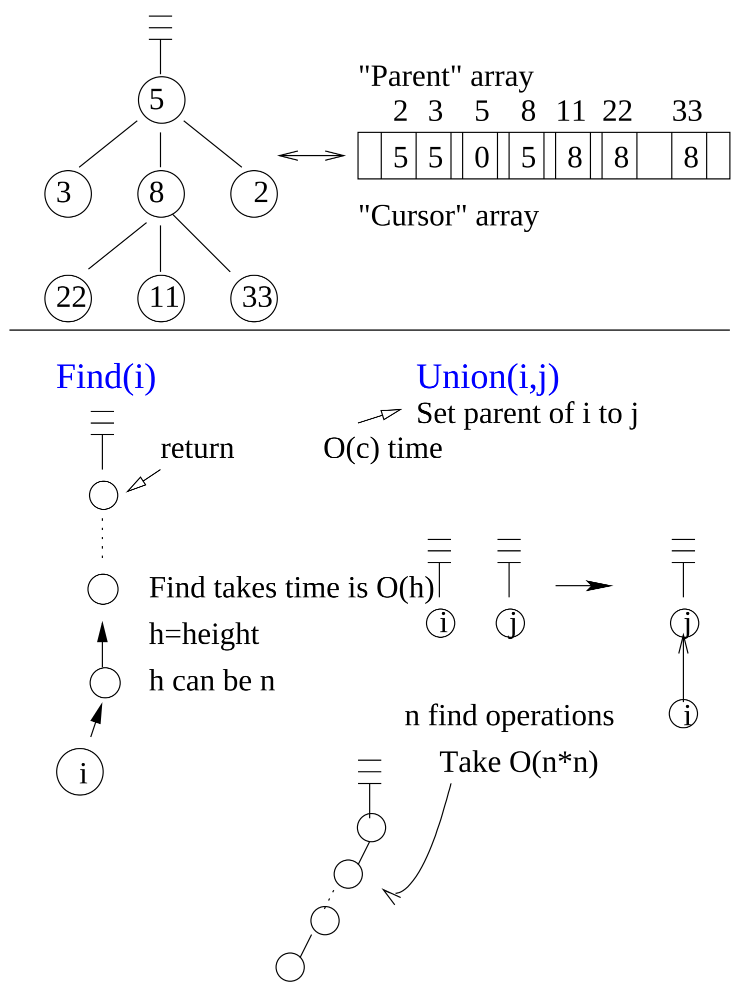
- Element j belongs to set A_j. Initially ...

	1	2		i		n
A	1	2		i		n

- Find(i):return(A[i]). $O(c)$ time
- Union(i,j): Change all i's to j in Array A (i and j are set names). Takes $O(n)$ time
- Cost of n operations: $O(n^2)$
- Cost of $m > n$ operations: $O(n^2 + m)$

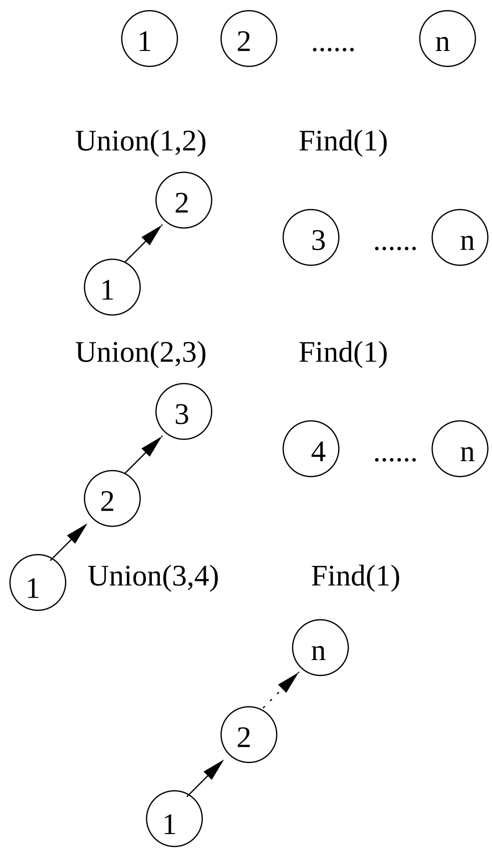
	1	2	3	4	5	6
U(2,3)	1	3	3	4	5	6
U(5,6)	1	3	3	4	6	6
U(1,4)	4	3	3	4	6	6
U(3,4)	4	4	4	4	6	6
U(4,6)	6	6	6	6	6	6

Reverse tree implementation

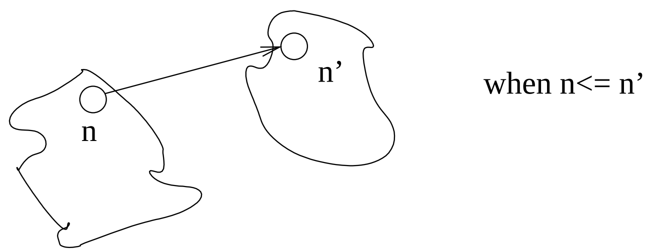


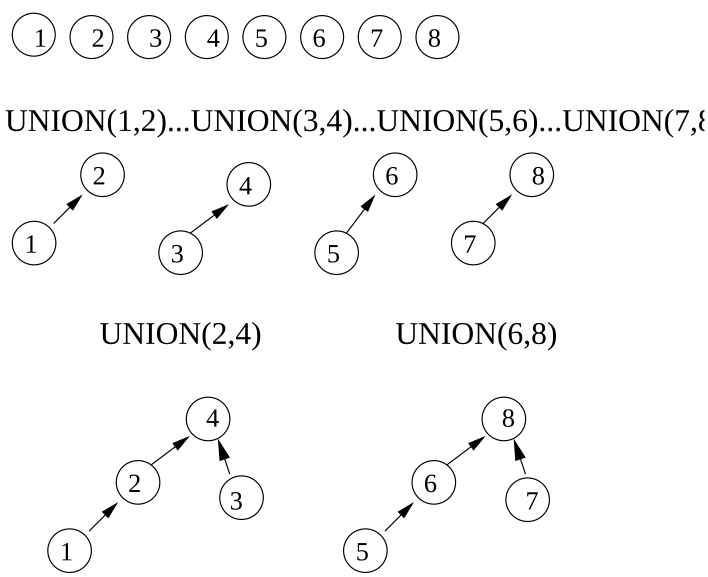
Example

Time complexity $\sum_{i=2}^{n} i$, which is $O(n^2)$

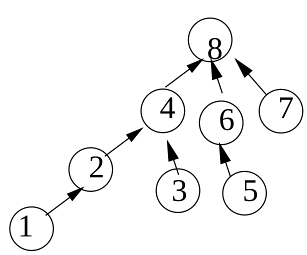


Weighted Union (union by size)





UNION(4,8)

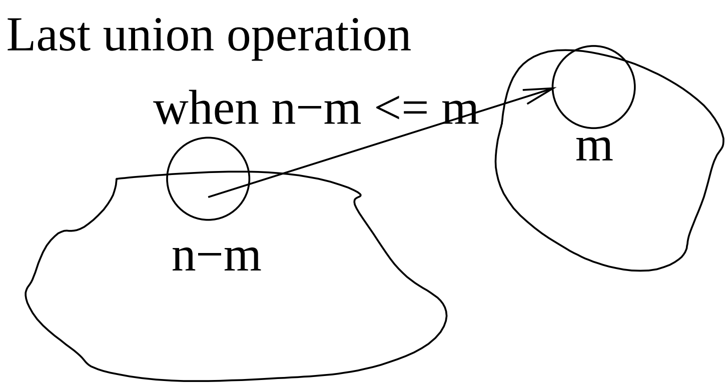


#	ht
1	1
2	2
4	3
8	4
......	
2^k	k+1

Theorem: If we have n nodes$\rightarrow$ h$\leq \lfloor \log_2 n \rfloor + 1$

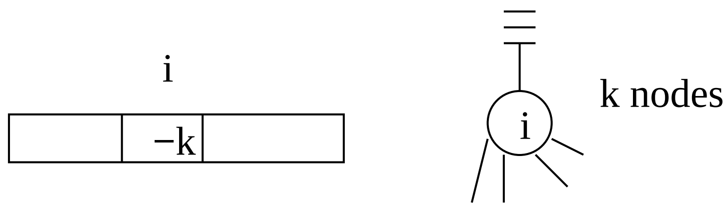
Proof:

- basis: $n = 1$, implies that $h = 1$.
- Ind Hypothesis: Assume true for $n - 1$
- Ind Step: Prove for $n \geq 2$,



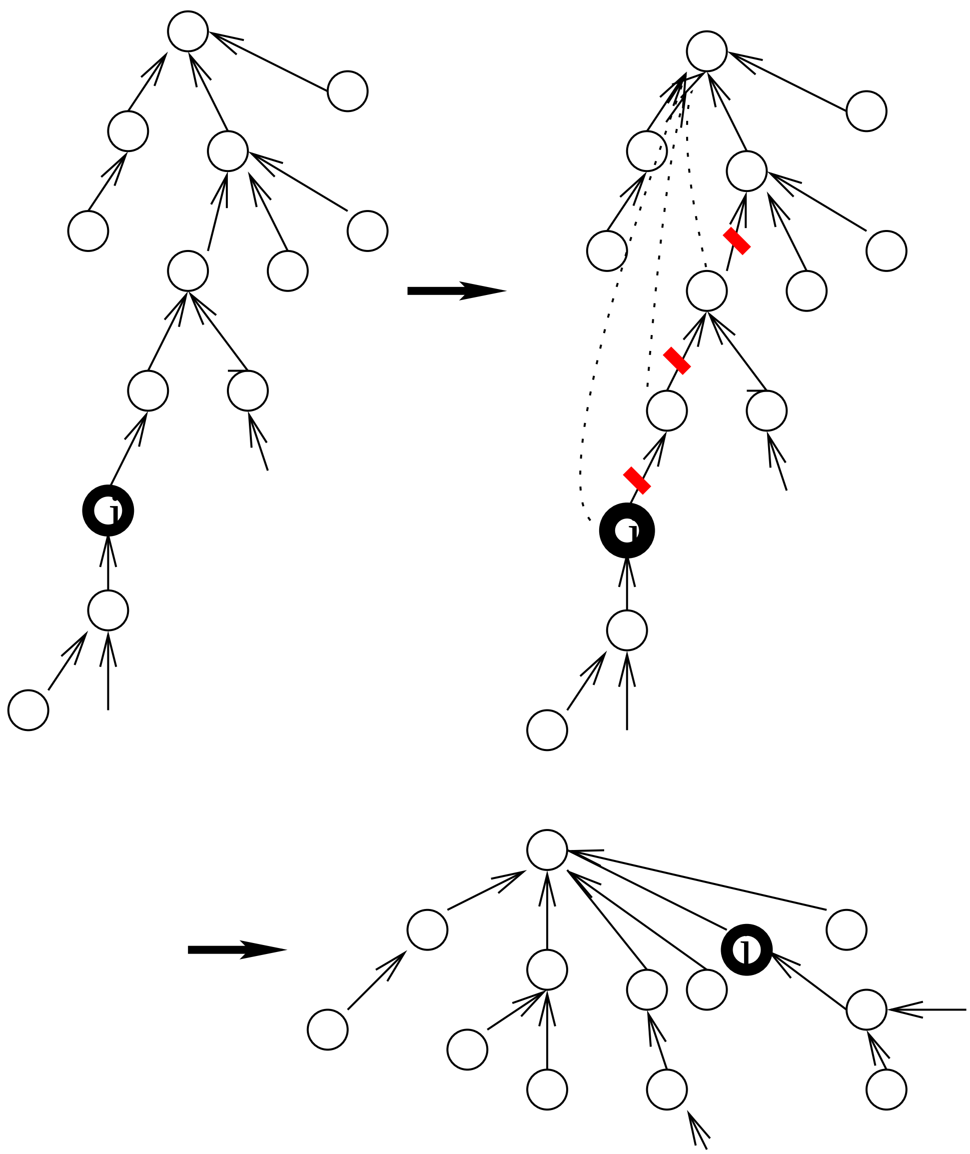
  - Height of "m"
    Since $m < n$ then by the ind hypothesis
    height of $m \leq \lfloor \log_2 m \rfloor + 1 \leq \lfloor \log_2 n \rfloor + 1$
  - Height of "$n - m$" after adding new root
    $\leq \lfloor \log_2 (n - m) \rfloor + 1 + 1 \leq \lfloor \log_2 (n/2) \rfloor + 2$
    $\leq \lfloor \log_2 n - 1 \rfloor + 2 \leq \lfloor \log_2 n \rfloor + 1$

- No additional space is required to store the number.



- N operations
  - Union takes $O(c)$ time.
  - Find takes $O(\log n)$ time.
  - Total time for n Union-Find operations is $O(n \log n)$,
- Also possible union by height but we do not do it because some changes (next slide).
- When does the worst case arise for weigthed union?

Path Compression



Next find(i) will not be as expensive

Analysis is quite complex(CS230A)

- M operations on N elements
  - Total time $O(M\alpha(M,N))$
  - $O(M\alpha(M,N))$ is a functional inverse of Ackermann's Function
  - $\alpha(M,N) \leq 4$ as long as $N \leq 2^{65536}$
  - Actually larger than a $20000 - digit$ number.