Maximum Independent Set Problem

Given an undirected graph $G = (V, E)$, find the largest integer $k$ such that there is an independent set $V'$ with cardinality $k$, i.e., $V' \subseteq V$ and no two vertices in $V'$ are adjacent (none of the edges in $E$ join two vertices in $V'$).

Associate $x_i$ with vertex $v_i$.

Restate problem: Find largest integer $k$ s.t. there is a vector $X$ with $\{0, 1\}$ entries, $\sum x_i = k$, and each edge $e = \{i, j\} \in E$ does not have both of its endpoints with value 1, i.e., $x_i + x_j \leq 1$. 
Integer Linear Prog. Formulation

ILP: Maximize  \[ \sum x_i \]
Subject to  \[ x_i + x_j \leq 1 \text{ for every } \{i, j\} \in E \]
\[ x_i = 0, \text{ or } 1, \text{ for } 1 \leq i \leq n. \]

In this case: Max  \[ x_1 + x_2 + x_3 + x_4 + x_5 \]
\[ x_1 + x_3 \leq 1, \quad x_1 + x_5 \leq 1, \quad x_2 + x_4 \leq 1 \]
\[ x_3 + x_4 \leq 1, \quad \text{and} \quad x_4 + x_5 \leq 1. \]
\[ x_i = 0 \text{ or } 1 \]
**Relaxed Problem**

Find largest real $k$ s.t. there is a vector $X$ with real entries values in $[0, 1]$ (i.e., $0 \leq x_i \leq 1$), $\sum x_i = k$, and each edge $e = \{e_i, e_j\} \in E$ satisfies $x_i + x_j \leq 1$.

**Linear Programming**

LP: Maximize $\sum x_i$

Subject to $x_i + x_j \leq 1$ for every $\{i, j\} \in E$

$0 \leq x_i \leq 1$, for $1 \leq i \leq n$. 
1. If $X'$ is a feasible solution to the maximum independent set problem, then $X'$ is also a feasible solution to the relaxed problem simply because $\{0, 1\} \subseteq [0, 1]$. Converse is not true.

2. Because of (1), $\text{OPT}(\text{ILP}) \leq \text{OPT}(\text{LP})$.

BIG GAP PROBLEM: $\text{OPT}(\text{ILP})$ may be too small compared to $\text{OPT}(\text{LP})$. Consider $G$ being the complete graph on $n$ vertices. $\text{OPT}(\text{ILP}) = 1$, but $\text{OPT}(\text{LP}) \geq n/2$. $\text{OPT}(\text{LP}) = n/2$ when $x_i = 1/2$ for $1 \leq i \leq n$. The resulting approximation algorithm using this technique is not bounded by any constant.
What sort of rounding could one use?

If we round up the values greater than or equal to 0.5, then it will not work. For example, if $x_i = x_j = 0.5$, then the constraint $x_i + x_j \leq 1$ is satisfied before the rounding, but not after the rounding.

If we round up the values greater than 0.5, then we also run into problems.

Suppose that we have a graph on $2n$ vertices labeled 1, 2, ..., 2n and the edges are $(i, n+i)$ for $1 \leq i \leq n$.

An optimum independent set has $n/2$ vertices. Therefore, $\text{OPT(ILP)} = n/2$.

An optimum solution to the corresponding LP problem has $x_i = 0.5$ for all $i$. The rounding has zero vertices in the independent set. So the rounded solution is far from optimal.