Part HW4a. This part does not involve actual programming and you are asked to design a parallelized solution for the heat distribution problem, partially through class discussion. We discuss the related solutions in a general context during May 15 lecture time. Each group should try to develop a solution partially or completely before May 22, share results and raise questions on May 22 and 24 for further refinement.

Deadline: May 23 Friday midnight using "turnin HW4a@cs140 LastNameFirstNameFilename". No late submission is permitted.

Description: Consider the problem of a heat distribution in a square with dimensions 1 mile by 1 mile. The temperature in the square boundary is given and fixed as:

\[ u(0, y) = 0, u(x, 0) = 0, u(x, 1) = 200x, \text{ and } u(1, y) = 200y. \]

The steady-state temperatures at interior points satisfy the following Laplace equation.

\[ \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0 \]

We divide the square region into a grid of \((n + 2) \times (n + 2)\) with gap \(h\) at each axis and \(h = \frac{1}{n+1}\). At each point \((ih, jh)\), let \(u(ih, jh) = u_{i,j}\). The goal is to find the value of all points \(u_{i,j}\). Figure 1(a) shows a grid of \(9 \times 9\) and there are \(7 \times 7\) interior unknown points to be solved.

1) (2 points) Why are the following equations true approximately for all \(n \times n\) interior points based on the Laplace equation?

\[ 4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0 \]

2) (6 points) Choose \(n = 2\). There are 4 interior points with unknown temperature values. 2a) List 4 equations for these 4 unknown values and write the pseudo code using the Jacobi method to solve these equations.

2b) Write the sequential pseudo code based on the Gauss-Seidel method.

2c) The kernel computation of this Gauss-Seidel method at each iteration is a 2D loop program. Illustrate the data dependence pattern for this 2D loop program given a general \((n + 2) \times (n + 2)\) grid.

3) (4 points) From 2), you will see the dependence structure for the Gauss-Seidel method requires more complex synchronization. To simplify the dependence, one method proposed is called the red-black method which marks the
unknown variables using black and red color. Figure 1(b) shows a grid of 10 × 10 and there are 8 × 8 interior unknown points marked in red and black. A black variable only depends on red variables while a red variable only depends on black variables. Notice that white circles in the boundary have the fixed known temperature values. Following this method, we try a small example with \( n = 2 \) as follows.

3a) Let \( u_{11} = B_1, u_{21} = R_1, u_{12} = R_2, u_{22} = B_2 \). List 2 equations for all black points and 2 equations for red points.

3b) Write pseudo code as a Jacobi iteration to get new values for all black points and then another Jacobi iteration to get new values for all red points. What is the advantage of the red-black method compared to the original Jacobi method you have outlined in 2)?

4) (8 points) Generalize your red-black solution for a \( (n + 2) \times (n + 2) \) grid as follows.

4a) Write the pseudo SPMD code to solve the heat equations on a shared memory machine. Assume the cost of thread synchronization using a barrier is zero, each addition or multiplication costs \( \omega \), and there are \( t \) iterations executed for updating new values for each grid point. What is the approximated parallel time with \( p \) cores?

4b) Write the message-passing pseudo SPMD code on a distributed memory machine using \( p \) processes. Estimate the approximate space size that needs to be allocated at each process to hold data points for computation and for communication.