Answers to Exercises in Quantum Computation I

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Answer 1 (Analyzing Small Circuits)**.**
(a) The circuit consists of four sequential transformations of 2 qubits, with the corresponding matrices:

$$I \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$H \otimes H = \frac{1}{2}\begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix}$$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and again the matrix of $H \otimes H$.

The effect of this circuit on a 2 qubit state $|\psi\rangle$ is described by $(H\otimes H)(\text{CNOT})(H\otimes H)(I\otimes X)\,|\psi\rangle$, which shows why the transformation matrix of this circuit equals

$$M_{\text{circuit}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

As a result, the behavior of the circuit on the four classical basis states is:

$\lvert a,b\rangle$	$M_{\text{circuit}}\lvert a,b\rangle$
$\lvert 0,0\rangle$	$\lvert 1,1\rangle$
$\lvert 0,1\rangle$	$\lvert 0,0\rangle$
$\lvert 1,0\rangle$	$\lvert 0,1\rangle$
$\lvert 1,1\rangle$	$\lvert 1,0\rangle$

(b) This circuit has the following effect on the four basis states

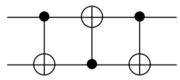
input	output
$\lvert 0,0\rangle$	$\lvert 0,0\rangle$
$\lvert 0,1\rangle$	$\lvert 0,1\rangle$
$\lvert 1,0\rangle$	$\lvert 1,0\rangle$
$\lvert 1,1\rangle$	$-\lvert 1,1\rangle$

,

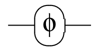
hence its matrix representation is

$$\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

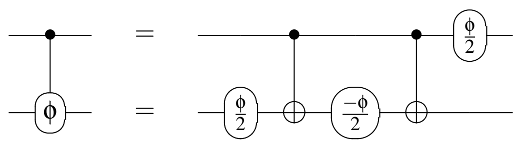
Answer 2 (Designing Small Circuits)**.**
(a) The following circuit implements the SWAP gate



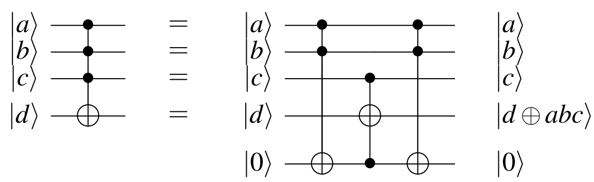
(b) From now on we denote the single qubit phase rotation gate by the circuit element

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \cong$$ 

We can implement the controlled phase rotation by



Answer 3 (Controlled NOTs)**.**
(a) It is not possible to implement a CCNOT gate using CNOT gates. Let $a,b,c \in \{0,1\}$ be the 3 bits input of a CNOT circuit. Describe the effect of the circuit on the third qubit as a $\bmod 2$ calculation such that CCNOT with its $|a,b,c\rangle \mapsto |a,b,c+ab\rangle$ is viewed as a second degree polynomial (because of the product term ab). As the CNot gate with its mapping $|a,b\rangle \mapsto |a,b\oplus a\rangle$ will only produce linear functions in a,b,c, a CNOT circuit is not capable of implementing a CCNOT gate.
(b) It is possible to implement a CCCNOT gate using CCNOT gates as the following circuit shows



Note how we drew the CCNOT gates here: The first CCNOT gate has as control bits $|a\rangle$ and $|b\rangle$, while $|c\rangle$ and $|d\rangle$ do not play a role, which is indicated by the solid control dots on the relevant wires.

Answer 4 (Unitarity)**.** Consider the effect of T on the vector $|\psi\rangle - |\phi\rangle$. Because $|\psi\rangle \neq |\phi\rangle$ we have $\|\,|\psi\rangle - |\phi\rangle\,\| \neq 0$. On the other hand, because T is linear, it must hold that $\|T(|\psi\rangle - |\phi\rangle)\| = \|\,|0,\dots,0\rangle - |0,\dots,0\rangle\,\| = 0$. Hence T is not norm preserving.

Acknowledgment: Again, the circuits in these exercises were drawn using the Q-circuit LaTeXpackage of Bryan Eastin and Steven T. Flammia.