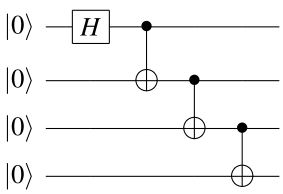
Answers to Exercises in Quantum Computation II

Wim van Dam
Department of Computer Science, University of California at Santa Barbara, Santa Barbara, CA 93106-5110, USA

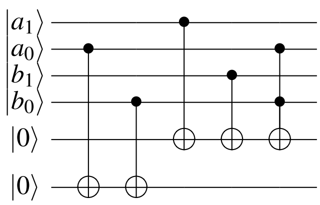
Answer 1. (The Effect of Pauli Gates). The following table describes the effect of the ?-gate on the output state.

?-gate	I	X	Y	Z
Output	$\|0,0\rangle$	$\|0,1\rangle$	$-\mathrm{i}\|1,1\rangle$	$\|1,0\rangle$

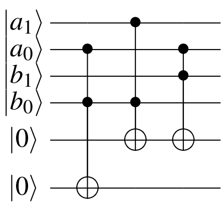
Answer 2. (Creating Correlated Quantum States). The following circuit creates the required superposition $\frac{1}{\sqrt{2}}(|0,0,0,0\rangle+|1,1,1,1\rangle)$.



Answer 3. (Implementing Modulo Calculations).
(a) Because $a+b=2(a_1+b_1)+(a_0+b_0)$, we have to implement the 6 bit operation $|a_1,a_0,b_1,b_0,0,0\rangle\mapsto|a_1,a_0,b_1,b_0,a_1+b_1+a_0b_0,a_0+b_0\rangle$, where the additions and multiplications are done $\bmod 2$.

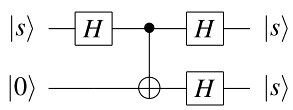


(b) Because $ab=4a_1b_1+2(a_1b_0+a_0b_1)+a_0b_0=2(a_1b_0+a_0b_1)+a_0b_0 \bmod 4$, we have to implement the 6 bit operation $|a_1,a_0,b_1,b_0,0,0\rangle\mapsto|a_1,a_0,b_1,b_0,a_1b_0+a_0b_1,a_0b_0\rangle$, where the additions and multiplications are done $\bmod 2$.



(c) Write the numbers as $a=\sum_{j=0}^{n-1}a_j2^j$ and $b=\sum_{j=0}^{n-1}b_j2^j$ such that $a+b=\sum_{j=0}^{n-1}(a_j+b_j)2^j$ and $ab=\sum_{j=0}^{n-1}(a_0b_j+\cdots+a_jb_0)2^j$. Use extra bits to implement the carry bits when adding or multiplying the numbers a and b. These techniques are well-known from traditional circuit complexity.

Answer 4. (Copying Qubits).
(a) Use the following circuit for all $s\in\{-,+\}$



(b) The effect of this circuit on the states $|b,0\rangle$ is $|0,0\rangle\mapsto\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle)$ and $|1,0\rangle\mapsto\frac{1}{\sqrt{2}}(|0,1\rangle+|1,0\rangle)$.
(c) (Proof by contradiction.) Let $U\in\mathbb{C}^{4\times 4}$ be the unitary matrix of the 2 qubit circuit. Because of the requirement $U:|0,0\rangle\mapsto|0,0\rangle$ the first column of the matrix U must have the form

$$\begin{pmatrix}1&*&*&*\\0&*&*&*\\0&*&*&*\\0&*&*&*\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Next, because of the other requirement $U:|+,0\rangle\mapsto|+,+\rangle$ the third column of U must have the form

$$\begin{pmatrix}1&*&\frac{1}{\sqrt{2}}-1&*\\0&*&\frac{1}{\sqrt{2}}&*\\0&*&\frac{1}{\sqrt{2}}&*\\0&*&\frac{1}{\sqrt{2}}&*\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Thus the first and the third column of U are not orthogonal, hence U is not unitary.

Alternatively, we can prove the nonunitarity of U as follows. We have $\langle 0,0|+,0\rangle=\frac{1}{\sqrt{2}}\neq\frac{1}{2}=\langle 0,0|+,+\rangle$, hence this U can not be inner-product preserving.

Answer 5. (Exploring an Unknown Function).
(a) Depending on the two values $f(0)$ and $f(1)$ the output states are as follows

$f(0)$	$f(1)$	output state
0	0	$\frac{1}{2}(+\|0\rangle+\|1\rangle)\otimes(\|0\rangle-\|1\rangle)$
0	1	$\frac{1}{2}(+\|0\rangle-\|1\rangle)\otimes(\|0\rangle-\|1\rangle)$
1	0	$\frac{1}{2}(-\|0\rangle+\|1\rangle)\otimes(\|0\rangle-\|1\rangle)$
1	1	$\frac{1}{2}(-\|0\rangle-\|1\rangle)\otimes(\|0\rangle-\|1\rangle)$

(b) If you apply a Hadamard on the first qubit and then measure it in the computational basis, then you will see a "0" if and only if $f(0)=f(1)$ and you will see a "1" if and only if $f(0)\neq f(1)$. Note that this quantum procedure can therefore decide the question $f(0)=f(1)$? with only one query to f.

Acknowledgment: Again, the circuits in these exercises were drawn using the Q-circuit LaTeXpackage of Bryan Eastin and Steven T. Flammia.