Answers to Exercises in Quantum Computation III

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Answer 1. (Deutsch-Jozsa on General Functions)

(a) —
(b) After the application of the $U_F$ gate the first $n$ qubits of the system are described by $\sum_x (-1)^{F(x)} |x\rangle / \sqrt{2^n}$, where the summation is over all strings $x \in \{0,1\}^n$. Applying $n$ Hadamard gates to this state gives

$$\rightarrow \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{F(x)} \sum_y (-1)^{x_1y_1 + \cdots + x_ny_n} |y\rangle.$$ 

As a result the inner product between this output state and $|0,\ldots,0\rangle$ is $2^{-n} \sum_x (-1)^{F(x)}$. Because $(-1)^{F(x)} = 1 - 2^F(x)$, this gives the probability of observing the all zeros state

$$\Pr(0\cdots0) = \frac{1}{2^n} \left( \sum_{x \in \{0,1\}^n} 1 - 2^F(x) \right)^2 = \left( 1 - 2 \frac{\sum_x F(x)}{2^n} \right)^2.$$ 

In other words, it depends quadratically on the Hamming weight $\sum_x F(x)$ of the string $F(0,\ldots,0),\ldots,F(1,\ldots,1)$. The probability is 1 if and only if $\sum_x F(x)$ is 0 or $2^n$; the probability is 0 if and only if $\sum_x F(x) = 2^{n-1}$.

Answer 2. (Preparation for Midterm) —

Answer 3. (One-Out-of-Four) Throughout the algorithm the 3rd qubit is used to implement the phase flip trick, and hence remains unchanged. After the first two steps, the first two qubits will be $\frac{1}{2} \sum_{x,y} |x,y\rangle - |s,r\rangle$ where the summation is over all bit combinations $(x,y) \in \{0,1\}^2$. Because we can write this also as $H \otimes H |0,0\rangle - |s,r\rangle$, we see that after step 3 the state equals $|0,0\rangle - H \otimes H |s,r\rangle$. Regardless of the values $s,r$, the amplitude of the $|0,0\rangle$ component of this state is $\frac{1}{2}$, hence after step 4 (with its $\frac{1}{2} |0,0\rangle \rightarrow - \frac{1}{2} |0,0\rangle$), the state will have changed to $- H \otimes H |s,r\rangle$. This shows that after the final transformation of step 5, the state of the system is $- |s,r\rangle$. Given the two unknown bits $s,r \in \{0,1\}$, the measurement will therefore reveal the values $s,r$ with 100% accuracy.

(a) This circuit searches a database of size 4 (encoded by $U_{x,r}$) in a single query. It is a small example of a quantum search algorithms that is more efficient than is possible classically.

Answer 4. (Summing Phases)

(a) Note that

$$\xi_n \cdot \left( \sum_{j=0}^{n-1} r_j^{ \xi_n } \right) = \sum_{j=1}^n r_j^{ \xi_n } = \sum_{j=0}^{n-1} r_j^{ \xi_n }.$$ 

Assuming $n \neq 1$, we have $\xi_n \neq 1$, which implies $\sum_{j=0}^{n-1} r_j^{ \xi_n } = 0$.

(b) Similarly, note that

$$\frac{1}{\zeta_n} \left( \sum_{j=0}^{p-1} r_j^{ \xi_n } \right) = \sum_{j=0}^{p-1} r_j^{ \xi_n }.$$ 

Hence, assuming that $q \neq n$, we have $\xi_n \neq 1$, which gives again $\sum_{j=0}^{p-1} r_j^{ \xi_n } = 0$. If $q = n$, then $p = 1$ and the summation becomes $\sum_{j=0}^{p-1} r_j^{ \xi_n } = 1$. (The degenerate case $n = 1$ obviously gives $\sum_{j=0}^{p-1} r_j^{ \xi_n } = p$.)

(c) For $n = 1$ the sum will be $m$. Otherwise, use the standard geometric sum equality

$$\sum_{j=0}^{m-1} \zeta_n^j = \frac{1 - \zeta_n^m}{1 - \zeta_n}.$$