Answers to Exercises in Quantum Computation V v2

Wim van Dam
Department of Computer Science, University of California at Santa Barbara, Santa Barbara, CA 93106-5110, USA

Answer 1. (Reading) —

Answer 2. (Towards Teleportation) (See Slides of Week 9 for more on these answers.)
(a) With $|q\rangle = \alpha|0\rangle + \beta|1\rangle$, the output state before the two measurements is

$$\begin{aligned}|\text{output}\rangle &= \tfrac{1}{2}(\alpha|0,00\rangle + \alpha|1,00\rangle + \alpha|0,11\rangle + \alpha|1,11\rangle + \beta|0,10\rangle - \beta|1,10\rangle + \beta|0,01\rangle - \beta|1,01\rangle)\\ &= \tfrac{1}{2}|00\rangle\otimes(\alpha|0\rangle+\beta|1\rangle) + \tfrac{1}{2}|01\rangle\otimes(\alpha|1\rangle+\beta|0\rangle) + \tfrac{1}{2}|10\rangle\otimes(\alpha|0\rangle-\beta|1\rangle) + \tfrac{1}{2}|11\rangle\otimes(\alpha|1\rangle-\beta|0\rangle).\end{aligned}$$

(b) The probability of measuring the outcome "00" on the first two qubits is $\frac{1}{4}$.
(c) The quantum state of the third qubit is $\alpha|0\rangle + \beta|1\rangle$ after the outcome "00" has been measured.
(d) For the four possible measurement outcomes we have the following cases

measurement outcome	probability	third qubit
00	$\frac{1}{4}$	$\alpha\|0\rangle + \beta\|1\rangle$
01	$\frac{1}{4}$	$\beta\|0\rangle + \alpha\|1\rangle$
10	$\frac{1}{4}$	$\alpha\|0\rangle - \beta\|1\rangle$
11	$\frac{1}{4}$	$-\beta\|0\rangle + \alpha\|1\rangle$

.

Answer 3. (Rewriting Entanglement)
(a) For $|q\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$, take the orthogonal qubit state $|q^\perp\rangle := \frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle$ such that

$$\begin{aligned}\frac{1}{\sqrt{2}}(|q,q\rangle + |q^\perp,q^\perp\rangle) &= \frac{1}{\sqrt{2}}\left(\frac{9}{25}|00\rangle + \frac{12}{25}|01\rangle + \frac{12}{25}|10\rangle + \frac{16}{25}|11\rangle + \frac{16}{25}|00\rangle - \frac{12}{25}|01\rangle - \frac{12}{25}|10\rangle + \frac{9}{25}|11\rangle\right)\\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\\ &= |\text{EPR}\rangle.\end{aligned}$$

(b) There is no qubit state $|q^\perp\rangle$ (with $|q\rangle = (|0\rangle + \mathrm{i}|1\rangle)/\sqrt{2}$) such that $(|qq\rangle + |q^\perp q^\perp\rangle)/\sqrt{2}$. Proof: Let $|q^\perp\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha,\beta\in\mathbb{C}$, then

$$\frac{1}{\sqrt{2}}(|q,q\rangle + |q^\perp,q^\perp\rangle) = \frac{1}{\sqrt{2}}\left(\tfrac{1}{2}|00\rangle + \tfrac{\mathrm{i}}{2}|01\rangle + \tfrac{\mathrm{i}}{2}|10\rangle - \tfrac{1}{2}|11\rangle + \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle\right).$$

For this state to equal $(|00\rangle + |11\rangle)/\sqrt{2}$ it most hold that $\beta^2 - \frac{1}{2} = 1$ and hence that $|\beta| = \sqrt{\frac{3}{2}}$, which contradicts the requirement that $|q^\perp\rangle = \alpha|0\rangle + \beta|1\rangle$ is a proper (normalized) qubit state.
(c) If $\alpha,\beta\in\mathbb{R}$, define $|q^\perp\rangle = \pm(\beta|0\rangle - \alpha|1\rangle)$ such that

$$\begin{aligned}\frac{1}{\sqrt{2}}(|q,q\rangle + |q^\perp,q^\perp\rangle) &= \frac{1}{\sqrt{2}}(\alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle + \beta^2|00\rangle - \beta\alpha|01\rangle - \alpha\beta|10\rangle + \alpha^2|11\rangle)\\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\\ &= |\text{EPR}\rangle,\end{aligned}$$

because, by normalization, we have $\alpha^2 + \beta^2 = 1$.
If α or $\beta \notin \mathbb{R}$, there is no such $|q^\perp\rangle$. Proof: Define $|q^\perp\rangle = \gamma|0\rangle + \delta|1\rangle$ such that

$$\begin{aligned}\frac{1}{\sqrt{2}}(|q,q\rangle + |q^\perp,q^\perp\rangle) &= \frac{1}{\sqrt{2}}(\alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle + \gamma^2|00\rangle + \gamma\delta|01\rangle + \gamma\delta|10\rangle + \delta^2|11\rangle)\\ &= \frac{1}{\sqrt{2}}\left((\alpha^2+\gamma^2)|00\rangle + (\alpha\beta+\gamma\delta)|01\rangle + (\alpha\beta+\gamma\delta)|10\rangle + (\beta^2+\delta^2)|11\rangle\right).\end{aligned}$$

For this to be the state $|\mathrm{EPR}\rangle$, the following equations must hold

$$\alpha^2+\gamma^2=1, \qquad \beta^2+\delta^2=1, \qquad \alpha\beta+\gamma\delta=0,$$

and by the normalization restriction $|\alpha|^2+|\beta|^2=|\gamma|^2+|\delta|^2=1$. Here we will show that this only possible if $\alpha,\beta,\gamma,\delta\in\mathbb{R}$. By adding the first two equations we get $(\alpha^2+\beta^2)+(\gamma^2+\delta^2)=2$. Now, because of the triangle inequality $|x+y|\leq|x|+|y|$ for $x,y\in\mathbb{C}$, we see that we must have $2=|(\alpha^2+\beta^2)+(\gamma^2+\delta^2)|\leq|\alpha^2+\beta^2|+|\gamma^2+\delta^2|\leq|\alpha^2|+|\beta^2|+|\gamma^2|+|\delta^2|=2$, and hence it must hold that $|\alpha^2+\beta^2|=|\gamma^2+\delta^2|=1$. By $(\alpha^2+\beta^2)+(\gamma^2+\delta^2)=2$ this implies $\alpha^2+\beta^2=\gamma^2+\delta^2=1$. With $|\alpha^2|+|\beta^2|=|\gamma^2|+|\delta^2|=1$ this is only possible if $\alpha,\beta,\gamma,\delta\in\mathbb{R}$. (We use here the fact that $(a+b\mathrm{i})^2+(c+d\mathrm{i})^2=a^2+2ab\mathrm{i}-b^2+c^2+2cd\mathrm{i}-d^2=1$ in combination with $a^2+b^2=c^2+d^2$ implies $b,d=0$.)

(d) For general $|q\rangle=\alpha|0\rangle+\beta|1\rangle$ and its orthogonal dual $|q^\perp\rangle=\beta^*|0\rangle-\alpha^*|1\rangle$ define the qubit state $|s\rangle=\alpha^*|0\rangle+\beta^*|1\rangle$ and its orthogonal dual $|s^\perp\rangle=\beta|0\rangle-\alpha|1\rangle$ such that

$$\begin{aligned}\frac{1}{\sqrt{2}}(|q,s\rangle+\left|q^\perp,s^\perp\right\rangle) &= \frac{1}{\sqrt{2}}(\alpha\alpha^*|00\rangle+\alpha\beta^*|01\rangle+\beta\alpha^*|10\rangle+\beta\beta^*|11\rangle+\beta^*\beta|00\rangle-\beta^*\alpha|01\rangle-\alpha^*\beta|10\rangle+\alpha^*\alpha|11\rangle)\\ &= \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\\ &= |\mathrm{EPR}\rangle,\end{aligned}$$

because, by normalization, we have $\alpha\alpha^*+\beta\beta^*=1$.