Question 1. (The Effect of Pauli Gates). Consider the following 2 qubit circuit:

\[
\begin{array}{c}
|0\rangle \\
|0\rangle
\end{array}
\begin{array}{cccc}
H & ? & H & \\
\uparrow & \downarrow & \downarrow & \downarrow
\end{array}
\begin{array}{c}
|0\rangle \\
|0\rangle
\end{array}
\]

where the ?-gate can be one of the Pauli matrices \{I, X, Y, Z\}. Calculate what the output of this quantum circuit will be depending on the choice for the ?-gate.

Question 2. (Creating Correlated Quantum States). Describe a 4 qubit circuit that from the input \(|0, 0, 0, 0\rangle\) produces the state \(\frac{1}{\sqrt{2}}(|0, 0, 0, 0\rangle + |1, 1, 1, 1\rangle)\).

Question 3. (Implementing Modulo Calculations). We are going to implement mod 4 calculations in a unitary (and hence reversible) way. Let \(a = (a_1, a_0)\) and \(b = (b_1, b_0)\) represent two numbers \(\in \mathbb{Z}_4\) according to \(a = 2a_1 + a_0\) and \(b = 2b_1 + b_0\). You are allowed to use all the standard gates.

(a) Describe a small 6 qubit circuit that implements the addition transformation \(|a, b, 0\rangle \mapsto |a, b, a + b \mod 4\rangle\).

(b) Describe a small qubit circuit that implements the multiplication transformation \(|a, b, 0\rangle \mapsto |a, b, ab \mod 4\rangle\).

(c) Contemplate how well you can generalize this to \(n\) bits numbers \(a, b \in \mathbb{Z}_{2^n}\).

Question 4. (Copying Qubits). Notation: We define the qubit states \(|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\) and \(|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\).

(a) The CNOT gate copies the bit value \(b\) in the operation \(|b, 0\rangle \mapsto |b, b\rangle\) for all \(b \in \{0, 1\}\). Describe a quantum circuit that copies the states \(|+, -\rangle\). That is: give a circuit that implements the mapping \(|s, 0\rangle \mapsto |s, s\rangle\) for all \(s \in \{+, -\}\).

(b) Describe the effect of the circuit of the previous question on the states \(|b, 0\rangle\) for all \(b \in \{0, 1\}\).

(c) Prove that there exists no quantum circuit that implements a 2 qubit operation with the mappings \(|0, 0\rangle \mapsto |0, 0\rangle\) and \(|+, 0\rangle \mapsto |+, +\rangle\). If possible, try to find more than one proof.

Question 5. (Exploring an Unknown Function). Take a Boolean function \(f : \{0, 1\} \rightarrow \{0, 1\}\) and define its 2 qubit unitary implementation \(U_f : |a, b\rangle \mapsto |a, b \oplus f(a)\rangle\).

(a) Apply this \(U_f\) to the superposition \(\frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)\) and see what happens to the two qubits depending on the function values \(f(0)\) and \(f(1)\).

(b) Ponder what you could do with the result of the previous question if you were allowed to apply some Hadamard gates after the application of \(U_f\).

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