Mathematics of Quantum Computation III

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Complete Measurements: Consider a quantum state $|\psi\rangle$ defined over the set of basis states $\mathcal{X}$

$$|\psi\rangle = \sum_{x\in\mathcal{X}} \alpha_x |x\rangle.$$

If we measure the this state in the $\mathcal{X}$-basis, that is according to the vectors $\{|x\rangle : x \in \mathcal{X}\}$, then with probability $|\alpha_x^2|$ we observe the outcome $x \in \mathcal{X}$ and the state collapses according to

$$\sum_{x\in\mathcal{X}} \alpha_x |x\rangle \mapsto |x\rangle .$$

Partial Measurements: Consider a quantum state $|\psi\rangle$ defined over the set of basis states $\mathcal{X}\times\mathcal{Y}$

$$|\psi\rangle = \sum_{x\in\mathcal{X},y\in\mathcal{Y}} \alpha_{x,y} |x,y\rangle.$$

If we measure the $\mathcal{Y}$-part of this state, then we will observe one of the possibilities $y \in \mathcal{Y}$, and the state $|\psi\rangle$ will collapse accordingly. Quantitatively, the probability of measuring y equals

$$\Pr(y|\psi) = \sum_{x\in\mathcal{X}} |\alpha_{x,y}^2|$$

and the state changes as

$$\sum_{x\in\mathcal{X},y\in\mathcal{Y}} \alpha_{x,y} |x,y\rangle \mapsto \frac{1}{\sqrt{\Pr(y|\psi)}} \sum_{x\in\mathcal{X}} \alpha_{x,y} |x,y\rangle.$$

Note that the outcome state is properly normalized again.

An alternative description of the effect of a partial measurement is the following. Describe the quantum state ψ by the superposition

$$|\psi\rangle = \sum_{y\in\mathcal{Y}} \beta_y |\phi_y, y\rangle.$$

Then the probability of measuring outcome $y \in \mathcal{Y}$ is simply $|\beta_y^2|$, and the induced collapse is $|\psi\rangle \mapsto |\phi_y, y\rangle$. The connection between these two descriptions is given by the equalities

$$\beta_y |\phi_y\rangle = \sum_{x\in\mathcal{X}} \alpha_{x,y} |x\rangle \text{ and } \beta_y = \sqrt{\sum_{x\in\mathcal{X}} |\alpha_{x,y}^2|}.$$

for all $y \in \mathcal{Y}$.

Note also that it does not matter whether we first measure the $\mathcal{Y}$ part of $|\psi\rangle$ and then the $\mathcal{X}$-part, or first the $\mathcal{X}$-part and then the $\mathcal{Y}$-part, or if we perform one complete measurement over $\mathcal{X}\times\mathcal{Y}$.

Two Qubit Example: Consider a Boolean measurement on the second qubit of the 2 qubit state

$$\frac{1}{\sqrt{3}}(|0,0\rangle + |0,1\rangle - |1,1\rangle).$$

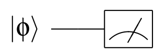
With probability $\frac{1}{3}$ the outcome of the measurement will be "0" after which the state has changed into $|0,0\rangle$. With probability $\frac{2}{3}$ the outcome of the measurement will be "1" after which the state has changed into

$$\frac{1}{\sqrt{2}}(|0,1\rangle - |1,1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\otimes|1\rangle .$$

Periodic State Example: This example is relevant to understand Shor's factoring algorithm. Let the set of basis states be of size NM and labeled by $\{0,1,\ldots,MN-1\}$. Consider a function $F : \{0,\ldots,NM-1\} \to \mathcal{Y}$ that has period M in the sense that $F(x) = F(y)$ if and only if $x - y = 0 \bmod M$. Because of the "if and only if" there are M different output values $F(x)$, and each value occurs N times in the sequence $F(0),\ldots,F(NM-1)$. Consider now what happens if we measure the $\mathcal{Y}$-register of the uniform superposition of F values $\sum_{x=0}^{NM-1} |x,F(x)\rangle/\sqrt{NM}$. With probability $1/M$ we will measure one of the M unique $y \in \mathcal{Y}$ values. Let $z \in \{0,\ldots,M-1\}$ be the unique such that $F(z) = y$, then, by the periodicity requirement, we also have $y = F(z) = F(z+M) = F(z+2M) = \cdots = F(z+(N-1)M)$, hence the state collapses according to

$$\frac{1}{\sqrt{NM}} \sum_{x=0}^{NM-1} |x,F(x)\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{\lambda=0}^{N-1} |z+\lambda M, F(z)\rangle.$$

Circuit Notation: A measurement of a single qubit $|\phi\rangle$ in the computational basis can be depicted by a 'meter'



[The circuits in these exercises were drawn using the Q-circuit LaTeXpackage of Bryan Eastin and Steven T. Flammia.]

Rotated Measurements: If we precede a standard qubit measurement by a Hadamard gate H, we can also say that we measure the qubit in the basis $\{|+\rangle,|-\rangle\} = \{\frac{|0+|1\rangle\rangle}{\sqrt{2}}, \frac{|0-|1\rangle\rangle}{\sqrt{2}}\}$. The reason for this terminology should be obvious. Given a qubit $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, the probability of measuring a "0" on $H|\phi\rangle$ equals $|\langle 0|H|\phi\rangle|^2 = |\langle +|\phi\rangle|^2$, while the probability of measuring a "1" on $H|\phi\rangle$ equals $|\langle 1|H|\phi\rangle|^2 = |\langle -|\phi\rangle|^2$.

In general, if we apply an inverse unitary rotation $U^\dagger$ to a quantum state $|\psi\rangle$ before measuring it in the computational basis, then we can consider this also as measurement of $|\psi\rangle$ in the rotated basis $\{U|0,\ldots,0\rangle,\ldots,U|1,\ldots,1\rangle\}$.