Administrivia

• Who has the book already?

• Office hours: Wednesday 13:30–15:30, otherwise by appointment.

• Midterm: 1/3, Final Project 2/3

• From now on: bring pen & paper: We will do calculations in class

• Extra notes will be posted before Thursday.

• Questions?
Loose Ends

What about two slit interference of bullets?

The wave length of a bullet is $\lambda \approx \frac{h}{mv}$ with Planck’s constant $h \approx 6.6 \times 10^{-34}$ J s, and $mv$ the mass $\times$ speed of the bullet. Take $m = 0.004$ kg and $v = 1000$ m/s, then $\lambda \approx 1.65 \times 10^{-34}$ meter.

This means that the distance between the slits has to be of the order of $10^3 \lambda \approx 10^{-31}$ m to have a noticeable effect.
“Painless learning” about quantum physics:

“Introducing Quantum Theory”,
J.P. McEvoy & O. Zarate ($10).

“QED: The Strange Theory of Light and Matter”,
R.P. Feynman ($11).
This Week

Mathematics of Quantum Mechanics:
• (Finite) Hilbert space formalism: vectors, lengths, inner products, tensor products.
• Finite dimensional unitary transformations.
• Projection Operators.

Circuit Model of Quantum Computation:
• Small dimensional unitary transformations as elementary quantum gates.
• Examples of important gates.
• Composing quantum gates into quantum circuits.
• Examples of simple circuits.
Quantum Mechanics

A system with $D$ basis states is in a superposition of all these states, which we can label by $\{1,\ldots,D\}$.

Associated with each state is a complex valued amplitude; the overall state is a vector $(\alpha_1,\ldots,\alpha_D) \in \mathbb{C}^D$.

The probability of observing state $j$ is $|\alpha_j|^2$.

When combining states/events you have to add or multiply the amplitudes involved.

Examples of Interference:
Constructive: $\alpha_1=\frac{1}{2}$, $\alpha_2=\frac{1}{2}$, such that $|\alpha_1+\alpha_2|^2 = 1$
Destructive: $\alpha_1=\frac{1}{2}$, $\alpha_2=-\frac{1}{2}$, such that $|\alpha_1+\alpha_2|^2 = 0$

(Probabilities are similar but with $\mathbb{R}$ instead of $\mathbb{C}$.)
Quantum Bits (Qubits)

- A single quantum bit is a linear combination of a two level quantum system: \{“zero”, “one”\}.

- Hence we represent that state of a qubit by a two dimensional vector \((\alpha, \beta) \in \mathbb{C}^2\).

- When observing the qubit, we see “0” with probability \(|\alpha|^2\), and “1” with probability \(|\beta|^2\).
- Normalization: \(|\alpha|^2 + |\beta|^2 = 1\).

- Examples: “zero” = \((1,0)\), “one” = \((0,1)\), uniform superposition = \((1/\sqrt{2}, 1/\sqrt{2})\), another superposition = \((1/\sqrt{2}, i/\sqrt{2})\)
Quantum Registers

• A string of \( n \) qubits has \( 2^n \) different basis states \(|x\rangle\) with \( x \in \{0,1\}^n \). The state of the quantum register \(|\psi\rangle\) has thus \( N=2^n \) complex amplitudes. In ket notation:

\[
|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle \in \mathbb{C}^{2^n}
\]

• \(|\psi\rangle\) is a column vector, with \( \alpha_x \) in alphabetical order.
• The probability of observing \( x \in \{0,1\}^n \) is \( |\alpha_x|^2 \).
• The amplitudes have to obey the normalization restriction:

\[
\sum_{x \in \{0,1\}^n} |\alpha_x|^2 = 1
\]

• What can we do with such a state?
Measuring is Disturbing

• If we measure the quantum state $|\psi\rangle$ in the computational basis $\{0,1\}^n$, then we will measure the outcome $x \in \{0,1\}^n$ with probability $|\alpha_x|^2$.

• For the rest, this outcome is fundamentally random. (Quantum physics predicts probabilities, not events.)

• Afterwards, the state has ‘collapsed’ according to the observed outcome: $|\psi\rangle \mapsto |x\rangle$, which is irreversible: all the prior amplitude values $\alpha_y$ are lost.
Time Evolution

• Given a quantum register $|\psi\rangle$, what else can we do besides measuring it? Answer: rotating it by $T$.

• Remember that $|\psi\rangle$ is a length one vector and if we change it, the outcome $|\psi'\rangle = T|\psi\rangle$ should also be length one: “$T$ is a norm preserving transformation”.

• Experiments show that QM is linear: $T$ has to be linear.

• Hence, if $T$ acts on a $D$-dimensional state space, then $T$ can be described by a $D \times D$ matrix $T \in \mathbb{C}^{D \times D}$.

• “$T$ is norm-preserving: $T$ is a (unitary) rotation.”
Classical Qubit Transformations

- Some simple qubit (D=2) transformations:
  - Identity with \( \text{Id} \): \(|\psi\rangle \mapsto |\psi\rangle \) for all \( \psi \):
    \[
    \text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
    \]
  - NOT gate with \( \text{NOT} \): \(|0\rangle \mapsto |1\rangle \) and \( \text{NOT} : |1\rangle \mapsto |0\rangle \); by linearity we have \( \text{NOT} : \alpha|0\rangle + \beta|1\rangle \mapsto \alpha|1\rangle + \beta|0\rangle \)
  - Note that NOT is norm-preserving: If \( |\psi\rangle \) has norm one, then so has \( \text{NOT}|\psi\rangle \)
    \[
    \text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
    \]
  - Also note: \( \text{NOT} : (|0\rangle + |1\rangle)/\sqrt{2} \mapsto (|1\rangle + |0\rangle)/\sqrt{2} \): the uniform superposition remains unchanged.
Hadamard Transform

- Define the Hadamard transform:
  \[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

- We have for this \( H \):
  \[ |0\rangle \mapsto \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \]
  \[ |1\rangle \mapsto \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \]

- Note: \( H^2 = \text{Id} \).
  It changes classical bits into superpositions and vice versa.

- It sees the difference between the uniform superpositions \((|0\rangle + |1\rangle)/\sqrt{2}\) and \((|0\rangle - |1\rangle)/\sqrt{2}\).
Hadamard Norm Preserving?

• Is H a proper quantum transformation? Linear: Yes, by definition, it is a matrix. Norm preserving? Hmmm….

• We have: $H: \alpha |0\rangle + \beta |1\rangle \mapsto \frac{(\alpha + \beta)}{\sqrt{2}} |0\rangle + \frac{(\alpha - \beta)}{\sqrt{2}} |1\rangle$

• Q: If $|\alpha|^2 + |\beta|^2 = 1$, then also $\frac{1}{2}|\alpha + \beta|^2 + \frac{1}{2}|\alpha - \beta|^2 = 1$?

• Use complex conjugates*: $|\alpha|^2 = \alpha \cdot \alpha^*$.

• Answer: Yes.
Hadamard as a Quantum Gate

- Often we will apply the H gate to several qubits.
- Take the n-zeros state $|0,\ldots,0\rangle$ and perform in parallel n Hadamard gates to the zeros, as a circuit:

Starting with the all-zero state and with only n elementary qubit gates we can create a uniform superposition of $2^n$ states.

Typically, a quantum algorithm will start with this state, then it will work in “quantum parallel” on all states at the same time.

As an equation:

$$|0,\ldots,0\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$
Combining Qubits

If we have a qubit \( |x\rangle = (|0\rangle + |1\rangle)\sqrt{2} \), then 2 qubits \( |x\rangle \) give the state \( \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \).

Tensor product notation for combining states \( |x\rangle \in \mathbb{C}^N \) and \( |y\rangle \in \mathbb{C}^M \): \( |x\rangle \otimes |y\rangle = |x\rangle |y\rangle = |x,y\rangle \in \mathbb{C}^{NM} \).

Example for two qubits: \( (\alpha_0 |0\rangle + \alpha_1 |1\rangle) \otimes (\beta_0 |0\rangle + \beta_1 |1\rangle) \)

\[
= \alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle
\]

Note that we multiply the amplitudes of the states. Also note the exponential growth of the dimensions.
Two Hadamard Gates

What does this circuit do on \{00, 01, 10, 11\}?

\[
\begin{align*}
|x_1\rangle & \hspace{1cm} H \hspace{1cm} |\ldots\ldots\rangle \\
|x_2\rangle & \hspace{1cm} H \hspace{1cm} |\ldots\ldots\rangle 
\end{align*}
\]

What is the \(\mathbb{C}^{4\times 4}\) rotation matrix of this operation?

What is the effect of \(n\) parallel Hadamard gates?

How does the corresponding \(2^n\times 2^n\) matrix look like?