Answers to Exercises I in Quantum Computation

Wim van Dam
Departments of Computer Science and Physics
University of California, Santa Barbara
Santa Barbara, CA 93106-5110, USA

Answers to Exercises for the course “Quantum Computation and Quantum Information”, Spring 2007.

Question 1 (Proving equalities)**.**
(a) We have $|A\rangle \otimes |B\rangle = \sum_{i,j} \alpha_i\beta_j |i,j\rangle$, and hence its norm equals $\sum_{i,j} |\alpha_i\beta_j|^2 = (\sum_i |\alpha_i|^2)(\sum_j |\beta_j|^2)$, which is the product of the norms of $|A\rangle$ and $|B\rangle$.
(b) The vector is properly normalized if and only if its inner product equals 1. Hence we have the requirement: $1 = (p\langle A,B| + q\langle C,D|)(p|A,B\rangle + q|C,D\rangle) = p^2\langle A,B|A,B\rangle + q^2\langle C,D|C,D\rangle + pq(\langle A,B|C,D\rangle + \langle C,D|A,B\rangle) = p^2 + q^2 + 2pq \cdot \mathrm{Real}(\langle A,B|C,D\rangle)$, where $\mathrm{Real}(\alpha)$ is the real component of the complex number α. Note that we used the facts that $p,q \in \mathbb{R}$ and $\langle A,B|A,B\rangle = \langle C,D|C,D\rangle = 1$.
(c) By the amplitudes $(2/3, 2/3)$ we see that the vectors $|A,B\rangle$ and $|C,D\rangle$ are not mutually orthogonal (following the previous answer, it must hold that $\mathrm{Real}(\langle AB|CD\rangle) = 1/8$), hence we cannot assume that the measurement of C implies that the state has collapsed completely to $|C,D\rangle$. Instead, rewrite $|A\rangle = x|C\rangle + y\left|C^{\perp}\right\rangle$ with $x,y \in \mathbb{C}$ and $\left|C^{\perp}\right\rangle$ a state vector orthogonal to $|C\rangle$ (where $x = \langle C|A\rangle$). As a result, we can rewrite the overall state by $2/3\,|A,B\rangle + 2/3\,|C,D\rangle = 2x/3\,|C,B\rangle + 2y/3\left|C^{\perp},B\right\rangle + 2/3\,|C,D\rangle = 2/3\,|C\rangle \otimes (x|B\rangle + |D\rangle) + 2y/3\left|C^{\perp},B\right\rangle$. Hence if we measure “$C$” on the left side, then the right part of the state will collapse to the state vector proportional to $x|B\rangle + |D\rangle = \langle C|A\rangle\,|B\rangle + |D\rangle$.

If we write the normalized state as $z(\langle C|A\rangle\,|B\rangle + |D\rangle)$, then the normalization term $z \in \mathbb{R}$ can be calculated as follows. It must hold that $1/z^2 = (\langle A|C\rangle\,\langle B| + \langle D|)(\langle C|A\rangle\,|B\rangle + |D\rangle) = 1 + |\langle C|A\rangle|^2 + \langle A,B|C,D\rangle + \langle C,D|A,B\rangle = 1 + |\langle C|A\rangle|^2 + 2\mathrm{Real}(\langle A,B|C,D\rangle)$. Because $\mathrm{Real}(\langle A,B|C,D\rangle) = 1/8$, we can conclude (finally) that $1/z^2 = 5/4 + |\langle C|A\rangle|^2$, and hence that after measuring “C” the normalized state on the right equals $(\langle C|A\rangle\,|B\rangle + |D\rangle)/\sqrt{5/4 + |\langle C|A\rangle|^2}$.
(d) Take any pair of sets of vectors $\{\left|A_2^{\perp}\right\rangle,\dots,\left|A_n^{\perp}\right\rangle\}$ and $\{\left|C_2^{\perp}\right\rangle,\dots,\left|C_n^{\perp}\right\rangle\}$ such that both $\{|A\rangle,\left|A_2^{\perp}\right\rangle,\dots,\left|A_n^{\perp}\right\rangle\}$ and $\{|B\rangle,\left|B_2^{\perp}\right\rangle,\dots,\left|B_n^{\perp}\right\rangle\}$ are orthonormal bases for $\mathbb{C}^n$. Then $U = |C\rangle\langle A| + \sum_{i=2}^{n} \left|C_i^{\perp}\right\rangle\left\langle A_i^{\perp}\right|$ is a proper unitary matrix with indeed $U : |A\rangle \mapsto |C\rangle$.

Question 2 (Unitarity)**.** Consider the effect of T on the vector $|\psi\rangle - |\phi\rangle$. Because $|\psi\rangle \neq |\phi\rangle$ we have $\|\,|\psi\rangle - |\phi\rangle\,\| \neq 0$. On the other hand, because T is linear, it must hold that $\|T(|\psi\rangle - |\phi\rangle)\| = \|\,|0,\dots,0\rangle - |0,\dots,0\rangle\,\| = 0$. Hence T is not norm preserving.

Question 3 (Inner Products)**.**
(a)

$$\langle\psi|\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{\mathrm{i}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{1+\mathrm{i}}{4}.$$

(b) Because U is unitary, the vector $U|3\rangle$ has to have unit length and it has to be orthogonal to $U|1\rangle$ and $U|2\rangle$. Hence $U|3\rangle = \mathrm{e}^{\mathrm{i}\phi}(\frac{4}{5}|1\rangle - \frac{3}{5}|2\rangle)$ with $\phi \in [0, 2\pi)$.

Question 4 (Hadamards)**.**
(a) $\frac{1}{4}$.
(b) $\frac{1}{2^k}$.
(c) If $k < n$ the probability is 0; if $k = n$ then it is $\frac{1}{2^n}$.

Question 5 (Analyzing Small Circuits)**.**
(a) The circuit consists of four sequential transformations of 2 qubits, with the corresponding matrices:

$$I \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$H \otimes H = \frac{1}{2}\begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{pmatrix}$$

$$\textsc{CNot} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and again the matrix of $H \otimes H$.

The effect of this circuit on a 2 qubit state $|\psi\rangle$ is described by $(H \otimes H)(\textsc{CNot})(H \otimes H)(I \otimes X)|\psi\rangle$, which shows why the transformation matrix of this circuit equals

$$M_{\text{circuit}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

As a result, the behavior of the circuit on the four classical basis states is:

$\lvert a,b\rangle$	$M_{\text{circuit}}\lvert a,b\rangle$
$\lvert 0,0\rangle$	$\lvert 1,1\rangle$
$\lvert 0,1\rangle$	$\lvert 0,0\rangle$
$\lvert 1,0\rangle$	$\lvert 0,1\rangle$
$\lvert 1,1\rangle$	$\lvert 1,0\rangle$

(b) This circuit has the following effect on the four basis states

input	output
$\|0,0\rangle$	$\|0,0\rangle$
$\|0,1\rangle$	$\|0,1\rangle$
$\|1,0\rangle$	$\|1,0\rangle$
$\|1,1\rangle$	$-\|1,1\rangle$

,

hence its matrix representation is

$$\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Question 6 (Understanding Single Qubit Gates)**.**
(a) As we must have $U \cdot U^{\dagger} = I$, the two columns have to have norm 1, hence $|\alpha^2| + |\beta^2| = 1$. It always holds that the vectors (α, β) and $(\beta^*, -\alpha^*)$ have inner product 0, so this $|\alpha^2| + |\beta^2| = 1$ is also a sufficient condition.