

Problem 1

SOLUTION HW 2

PHYS 105A

①

$\vec{v}(t)$ = velocity of a particle

$|\vec{v}|$ = magnitude of \vec{v}

$\frac{d\vec{v}}{dt}$ = acceleration

Ⓐ We need to prove that $\frac{d\vec{v}}{dt} \perp \vec{v} \Rightarrow |\vec{v}| = \text{constant}$

We know

$$\frac{d\vec{v}}{dt} \perp \vec{v} \Rightarrow \frac{d\vec{v}}{dt} \cdot \vec{v} = 0$$

We consider $\frac{d}{dt} (\vec{v} \cdot \vec{v}) = \frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt}$

$$\frac{d}{dt} (\vec{v} \cdot \vec{v}) = 2 \frac{d\vec{v}}{dt} \cdot \vec{v}$$

$$\frac{d}{dt} (\vec{v} \cdot \vec{v}) = 0$$

$$\frac{d}{dt} |\vec{v}|^2 = 0 \Rightarrow |\vec{v}| \text{ constant.}$$

~~also~~

Ⓑ We need to prove that $|\vec{v}| = \text{constant} \Rightarrow \frac{d\vec{v}}{dt} \perp \vec{v}$

We know $|\vec{v}| = \text{constant}$

and also $\vec{v} \cdot \vec{v} = \text{constant}$

taking derivative $\frac{d}{dt} (\vec{v} \cdot \vec{v}) = 0$

$$\frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = 0$$

$$2 \frac{d\vec{v}}{dt} \cdot \vec{v} = 0$$

$$\vec{v} \cdot \vec{v} = 0$$

$$\text{so } \vec{v} = \vec{0}$$

$$\vec{v} + \frac{d\vec{v}}{dt} = \vec{0} \quad \text{with } \vec{v} = \vec{0} \quad \text{at } t = 0 \quad \text{d)}$$

$$\vec{v} = \vec{0} \quad \text{at } t = 0$$

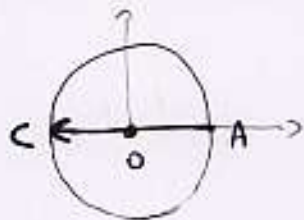
$$0 = \left(\frac{d\vec{v}}{dt} \right)_{t=0}$$

Problem 2

SOLUTION HW 2

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(a) The puck as seen from the ground travels in a straight line through the center.



It starts from the point A at $t=0$

$$\begin{cases} \phi(t) = 0 \\ r(t) = R - v_0 t \end{cases} \quad \text{with } v_0 \text{ constant speed}$$

It falls off at point C after $T = \frac{2R}{v_0}$

(b) Imagine an observer sitting on the rotating turntable near A. As seen from the ground he is traveling north with a speed ωR . As seen by the observer, the puck's initial velocity has a component ωR south.



$$\begin{cases} r(t) = R - v_0 t \\ \phi(t) = \phi(0) - \omega t = -\omega t \end{cases}$$

As the puck moves to a smaller radius r , the component of velocity ωr gets less. After $t = \frac{R}{v_0}$ the puck is on the center.



$$\dot{r}(t) = -v_0$$

$$\dot{\phi}(t) = -\omega r$$

Homework #2. Part 2. Solution.

Q3)

$$(a) \quad m\dot{V}_x = -bV_x, \quad m\dot{V}_y = mg - bV_y$$

$$\Rightarrow V_x(t) = V_x(0) e^{-t \cdot \frac{b}{m}} \\ = V_x(0) \left[1 - t \cdot \frac{b}{m} + \frac{b^2}{2m^2} t^2 + \dots \right]$$

$$\text{as } b \rightarrow 0, \quad V_x(t) \rightarrow V_x(0) \quad \text{--- ①}$$

$$V_y(t) = \frac{mg}{b} + \left(V_y(0) - \frac{mg}{b} \right) e^{-t \cdot \frac{b}{m}} \\ = \frac{mg}{b} + \left(V_y(0) - \frac{mg}{b} \right) \left(1 - t \cdot \frac{b}{m} + \frac{1}{2} \left(\frac{b}{m} \right)^2 t^2 + \dots \right)$$

$$= \frac{mg}{b} + \left(V_y(0) - \frac{mg}{b} \right) + \left(-\frac{b}{m} V_y(0) t + gt \right) + \dots$$

$$= \underbrace{\left(\frac{mg}{b} - \frac{mg}{b} \right)}_{O\left(\frac{1}{b}\right)} + \underbrace{\left(V_y(0) + gt \right)}_{O(1)} + \underbrace{\left(-\frac{b}{m} V_y(0) t - \frac{1}{2} g \left(\frac{b}{m} \right) t^2 \right)}_{O(b)} + \dots$$

$$= V_y(0) + gt + O(b)$$

$$\text{and as } b \rightarrow 0, \quad O(b) \rightarrow 0, \quad \Rightarrow \underline{\underline{\lim_{b \rightarrow 0} V_y(t) = V_y(0) + gt}}$$

(b) For the case of quadratic air resistance,
For simplicity, we assume the following eqn. of motion

$$\begin{cases} m\dot{V}_x = -cV_x^2 \\ m\dot{V}_y = mg - cV_y^2 \end{cases}$$

$$\Rightarrow V_x(t) = \frac{V_x(0)}{1 + \frac{cV_x(0)}{m} t} \quad \Rightarrow \quad \lim_{c \rightarrow 0} V_x(t) = V_x(0)$$

$$V_y(t) = \sqrt{\frac{mg}{c}} \tanh\left(g\sqrt{\frac{c}{mg}}t\right) \quad (V_y(0) = 0)$$

Since. $\tanh x = x - \frac{x^3}{3} + \dots$

$$V_y(t) = \sqrt{\frac{mg}{c}} \left(g\sqrt{\frac{c}{mg}}t - \frac{1}{3} \left(\sqrt{\frac{c}{m}g}\right)^3 t^3 + \dots \right)$$

$$= \underbrace{gt}_{O(t)} - \frac{1}{3} \underbrace{\frac{c}{m}g}_{O(c^1)} t^3 + \dots$$

$$O(c^n), \quad n \geq 2.$$

$$\Rightarrow \lim_{c \rightarrow 0} V_y(t) = \underline{gt}$$

(Q4) $F = m\dot{V} \Rightarrow F \cdot V = mV \cdot \dot{V} = \frac{m}{2} \frac{d}{dt}(V^2)$

$$= F \cdot \frac{dx}{dt}$$

integrate
 \Rightarrow

$$\int_{x_0}^x F \cdot dx = \frac{m}{2} \int_{V_0}^V V^2$$

$$\Rightarrow \underline{V^2 = V_0^2 + \frac{2}{m} \int_{x_0}^x F \cdot dx}$$

if $F(x)$ is constant,

then $\frac{m}{2} V^2 - \underbrace{\int_{x_0}^x F \cdot dx}_{F \cdot (x-x_0)} = \frac{m}{2} V_0^2$

or $\frac{m}{2} V^2 - \frac{m}{2} V_0^2 = F \cdot (x-x_0)$

$$\Delta T = \cancel{W} W$$

(change of kinetic energy) (Work done by the force)

Q5)

$$\begin{aligned} a) \quad \dot{V}_x &= \omega V_y & \Rightarrow \quad \ddot{V}_x &= \omega \dot{V}_y = \omega(-\omega V_x) \\ \dot{V}_y &= -\omega V_x & & \text{or } \ddot{V}_x + \omega^2 V_x = 0 \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} V_x &= A \sin \omega t + B \cos \omega t \\ V_y &= \frac{V_x}{\omega} = A \cos \omega t - B \sin \omega t \end{aligned}}$$

$$\begin{aligned} b) \quad \eta &= V_x + i V_y \Rightarrow \dot{\eta} = \omega(V_y - i V_x) = -i\omega \eta \\ \eta &= C \cdot e^{-i\omega t}, \quad C \Rightarrow \text{complex number} \end{aligned}$$

$$= V_x + i V_y = A \sin \omega t + B \cos \omega t + i(A \cos \omega t - B \sin \omega t)$$

$$= iA e^{-i\omega t} + B e^{-i\omega t} \quad (\because e^{-i\omega t} = \cos \omega t - i \sin \omega t)$$

$$= (B + iA) e^{-i\omega t}$$

So, if we choose
 $C = B + iA$

then
 $\eta = (B + iA) e^{-i\omega t}$
 $= V_x + i V_y$

Q6)

$$(a) \quad \frac{m dv}{v^3} = -\frac{c dt}{m}$$

$$\Rightarrow \int_{v_0}^v \frac{dv}{v^3} = -\frac{c}{m} t \quad \left. \vphantom{\int} \right\} \Rightarrow \frac{1}{v^2} = \frac{1}{v_0^2} + \frac{2c}{m} t$$
$$= -\frac{v^{-2}}{2} \Big|_{v_0}^v = -\frac{1}{2} \left(\frac{1}{v^2} - \frac{1}{v_0^2} \right)$$

$$\text{or } v = v_0 \cdot \frac{1}{\sqrt{1 + \frac{2c v_0^2}{m} t}}$$

$$(b) \quad x = \int_0^t \frac{V_0}{\sqrt{1 + \frac{2cV_0^2}{m}t}} dt$$

$$= V_0 \cdot \frac{m}{cV_0^2} \left[\sqrt{1 + \frac{2cV_0^2}{m}t} - 1 \right]$$

$$= \frac{m}{cV_0} \left[\sqrt{1 + \frac{2cV_0^2}{m}t} - 1 \right]$$