Question 1, 

(Maximizing area)

\[ A = \int y \, dx, \quad ds = \sqrt{dx^2 + dy^2}. \]

on \( dx = \sqrt{1 - y'^2} \, ds \), where \( y' = \frac{dy}{ds} \)

\[ y = \int y \sqrt{1 - y'^2} \, ds. \]

(here we regard \( y \) as

\[ f(y, y', s) \]

a function of \( s \)).

Since \( f \) does not depend on \( s \) explicitly, there exists

a first integral,

\[ f - y' \frac{df}{dy'} = \frac{y}{\sqrt{1 - y'^2}} = R \quad \text{(constant of motion)} \]

or

\[ y' = \sqrt{1 - y^2/R^2} \quad \Rightarrow \quad \frac{dy}{\sqrt{1 - y^2/R^2}} = ds \]

integrating both sides, we conclude that,

\[ \arcsin \left( \frac{y}{R} \right) = \frac{s}{R} \quad (\therefore \quad y(s = 0) = 0) \]

or \( y = R \sin \left( \frac{s}{R} \right) \), since \( y = 0 \) when \( s = 0 \),

we see that \( l/R = \pi \). (It is easy to see that

the other solutions, \( l/R = 2\pi, 3\pi, \ldots \), yield a smaller area.)

Finally,

\[ x = \int \sqrt{1 - y^2} \, ds = R \int \cos \left( \frac{s}{R} \right) \]

\( \Rightarrow \) \( 6(-R)^2 + y^2 = R^2 \) \( \therefore \) So the string must lie

on the semicircle.
Question 2.

The total length of a path is

\[ L = \int \sqrt{dx^2 + dy^2 + dz^2} \]

\[ = \int \sqrt{x'^2 + y'^2 + z'^2} \, du \]

\[ f(x, y, z, x', y', z', u) \]

Lagrange Eq. \( \Rightarrow \) Since \( \frac{df}{dx} = \frac{df}{dy} = \frac{df}{dz} = 0 \)

\[ \Rightarrow \frac{df}{dx'} = \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_1 \]

\[ \frac{df}{dy'} = \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_2 \]

\[ \frac{df}{dz'} = \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_3 \]

\[ \Rightarrow \text{solutions} \]

\[ x' : y' : z' = c_1 : c_2 : c_3 \]

or \( x' = c_1 g(u) \)

\[ y' = c_2 g(u) \]

\[ z' = c_3 g(u) \]

by re-defining the parameter as, \( t = \int dx' = \int du \cdot g(u) \)

\[ \frac{dx}{dt} = C_1, \quad \frac{dy}{dt} = C_2, \quad \frac{dz}{dt} = C_3 \]

\[ \Rightarrow \text{It is obvious from this form that the curve is a straight line.} \]
Question 3 
(Fermat's Principle)

\[ \overline{AP} = 2R \sin \left( \frac{\pi}{2} - \theta \right), \quad \overline{PB} = 2R \sin \left( \frac{\pi}{2} + \theta \right) \]

\[ \Rightarrow \overline{AP} + \overline{PB} = 4R \sin \left( \frac{\pi}{4} \right) \cos \frac{\theta}{2} = 2\sqrt{2} R \cos \frac{\theta}{2} \]

It is obvious from this form that the total distance is maximum when \( \theta = 0 \). Or, Light travel the longest route between two points, in this case.

Question 4.
The area of the surface of revolution is

\[ A = \int 2\pi y \, ds = 2\pi \int y \sqrt{1 + x'^2} \, dy \], where \( x' = \frac{dx}{dy} \),

and Euler-Lagrange equation reads,

since \( \frac{2x}{dx} = 0 \), \( \frac{2x}{dx} = \frac{y}{\sqrt{1 + x'^2}} = \frac{y}{y_0} \) (constant)

Hence, \( x' = \frac{y}{\sqrt{y^2 - y_0^2}} \). Using the substitution,

\[ y/y_0 = \cosh u \], you can integrate this to give

\[ x - x_0 = y_0 \arccosh \left( \frac{y}{y_0} \right) \] or \[ y = y_0 \sinh \left( \frac{x-x_0}{y_0} \right) \]
Questions (cycloid)

\[ x = a (\theta - \sin \theta) \]

\[ y = a (1 - \cos \theta) \]

(Note, - sign)

\[ ds = \sqrt{dx^2 + dy^2} = a \sqrt{2} \sqrt{(-a \sin \theta)} \, d\theta. \]

The speed of the cart is given by conservation of energy as:

\[ v = \sqrt{2g(y - y_0)} = \sqrt{2ga(a \cos \theta_0 - a \sin \theta_0)} \]

Therefore, the required time is

\[ t = \int_{\theta_0}^{\pi} \frac{a \sqrt{2} \sqrt{1-a \sin \theta}}{\sqrt{2ga(a \cos \theta_0 - a \sin \theta_0)}} \, d\theta \]

Make the substitution, \( \theta = \pi - 2x \) and use a couple of trigonometric identities (as \( 2x = 2a^2x - 1 \), \( 1 = 2\sin^2 x \)),

This becomes

\[ t = 2 \sqrt{\frac{a}{g}} \int_0^{a \cos x} \frac{\cos x}{\sqrt{\sin^2 \theta_0 - \sin^2 x}} \, dx \]

Finally, let \( \sin x = U \),

Then,

\[ t = 2 \sqrt{\frac{a}{g}} \int_0^{a \cos x} \frac{1}{\sqrt{U^2 - U^2}} \, dU = 2 \sqrt{\frac{a}{g}} \int_0^1 \frac{1}{\sqrt{1-U^2}} \, dU = 2 \sqrt{\frac{a}{g}} \left[ \arcsin U \right]_0^1 \]

\[ = \pi \sqrt{\frac{a}{g}} \]

b) The higher the starting point \( P_0 \), the further the car has to go, but the steeper the initial slope, the faster the car goes. On a cycloid, these two effects cancel each other.