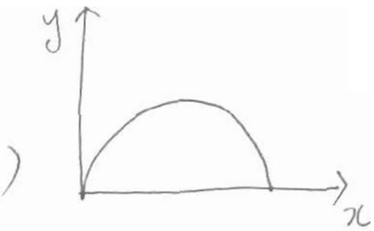
Phys 105A HW #7 Solutions

Question 1,

(Maximizing area)



$A = \int y\,dx$, $ds = \sqrt{dx^2 + dy^2}$, or $dx = \sqrt{1 - y'^2}\,ds$, where $y' = \dfrac{dy}{ds}$

area

then. $A = \int_0^l \underbrace{y\sqrt{1-y'^2}}_{f(y,y';s)}\,ds$. (here we regard y as a function of s)

Since f does not depend on s explicitly, there exists a first integral,

$$f - y'\frac{\partial f}{\partial y'} = \frac{y}{\sqrt{1-y'^2}} = R \text{ (constant of motion)}$$

or $\quad y' = \sqrt{1 - y^2/R^2} \iff \dfrac{dy}{\sqrt{1 - y^2/R^2}} = ds$

integrating both sides, we conclude that,

$$\arcsin(y/R) = s/R \quad (\because\ y(s=0) = 0)$$

or $\quad \underline{\underline{y = R\sin\left(\frac{s}{R}\right)}}$, since $y = 0$ when $s = l$,

we see that $l/R = \pi$. (It is easy to see that the other solutions, $l/R = 2\pi, 3\pi, \cdots$, yield a smaller area)

Finally, $x = \int \sqrt{1-y'^2}\,ds = R - R\cos(s/R)$

$\Rightarrow \underline{(x-R)^2 + y^2 = R^2}$. $\therefore$ So the string must lie on the semicircle.

Question 2.

total length of a path is $L = \int \sqrt{dx^2 + dy^2 + dz^2}$

$$= \int \underbrace{\sqrt{x'^2 + y'^2 + z'^2}}_{f(x,y,z,x',y',z',u)}\, du$$

Lagrange Eq $\Rightarrow$ since $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_1$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_2$$

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_3$$

$\Rightarrow$ solutions

$x' : y' : z' = C_1 : C_2 : C_3$

or $x' = C_1\, g(u)$

$y' = C_2\, g(u)$

$z' = C_3\, g(u)$

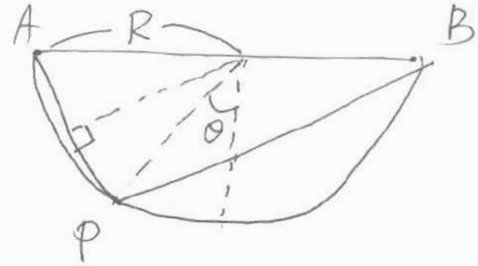
by re-defining the parameter as, $t = \int d\cancel{t} = \int du \cdot g(u)$

$$\frac{dx}{d\cancel{t}} = C_1, \quad \frac{dy}{dt} = C_2, \quad \frac{dz}{dt} = C_3$$

$\Rightarrow$ It is obvious from this form that the curve is a straight line.

Question 3

(Fermat's Principle)



$$\overline{AP} = 2R\sin\left(\frac{\frac{\pi}{2}-\theta}{2}\right), \quad \overline{PB} = 2R\sin\left(\frac{\frac{\pi}{2}+\theta}{2}\right)$$

$$\Rightarrow \overline{AP}+\overline{PB} = 4R\sin\frac{\pi}{4}\cos\frac{\theta}{2} = \underline{2\sqrt{2}R\cos\frac{\theta}{2}}$$

it is obvious from this form that the total distance is maximum when $\theta = 0$. Or, Light travel the longest route between two points, in this case.

Question 4.

The area of the surface of revolution is $A = \int 2\pi y\, ds$

$= 2\pi\int y\sqrt{1+x'^2}\,dy$, where $x' = \frac{dx}{dy}$,

and Euler-Lagrangian eq reads,

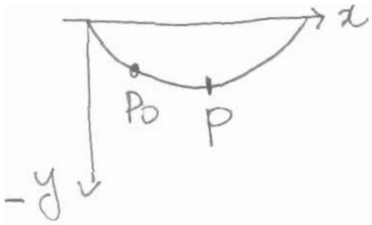
since $\frac{\partial \mathcal{L}}{\partial x} = 0$, $\quad \frac{\partial \mathcal{L}}{\partial x'} = \frac{yx'}{\sqrt{1+x'^2}} = y_0$ (constant)

Hence, $x' = \frac{y_0}{\sqrt{y^2-y_0^2}}$. Using the substitution,

$y/y_0 = \cosh u$, you can integrate this to give

$x - x_0 = y_0 \operatorname{arccosh}(y/y_0)$ or $\underline{y = y_0\cosh[(x-x_0)/y_0]}$

Question 5 (cycloid)



a) $x = a(\theta - \sin\theta)$, $y = -a(1-\cos\theta)$ } parametric eq of the cycloid.

(note, $-$ sign)

$$\Rightarrow ds = \sqrt{dx^2+dy^2} = a\sqrt{2(1-\cos\theta)}\, d\theta .$$

the speed of the cart is given by conservation of energy as,

$$V = \sqrt{2g(y_0-y)} = \sqrt{2ga(\cos\theta_0 - \cos\theta)}$$

therefore the required time is

$$t = \int_{P_0}^{P} \frac{ds}{v} = \int_{\theta_0}^{\pi} \frac{a\sqrt{2(1-\cos\theta)}}{\sqrt{2ga(\cos\theta_0-\cos\theta)}}\, d\theta$$

make the substitution, $\theta = \pi - 2\alpha$ and use a couple of trigometric identities ($\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$),

this becomes

$$t = 2\sqrt{\frac{a}{g}} \int_0^{\alpha_0} \frac{\cos\alpha}{\sqrt{\sin^2\alpha_0 - \sin^2\alpha}}\, d\alpha ,$$ Finally let $\sin\alpha = u$

then,

$$t = 2\sqrt{\frac{a}{g}} \int_0^{u_0} \frac{du}{\sqrt{u_0^2-u^2}} = 2\sqrt{\frac{a}{g}} \int_0^1 \frac{dv}{\sqrt{1-v^2}} = 2\sqrt{\frac{a}{g}} \Big[\arcsin V\Big]_0^1$$

$$= \pi\sqrt{\frac{a}{g}}$$

b) the higher the starting point Po, the further the car has to go, but the steeper the initial slope, the faster the car goes. On a cycloid, these two effects cancel each other.