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Midterm and Homework 5, Phys 105A

Question 1. A NONCONSERVATIVE FORCE

Consider a nonconservative force defined over the plane with the following (topological) property for the work done over the closed paths from 1 back to 1: $W(1 \to 1) = 0$ if the loop does not go around the origin O, $W(1 \to 1) = c$ if the loop goes around the origin O once in a clockwise fashion, $W(1 \to 1) = -c$ if the loop goes around the origin O once in an anti-clockwise fashion, and so on. In other words $W(1 \to 1)/c$ counts how many times the path went around O clockwise.

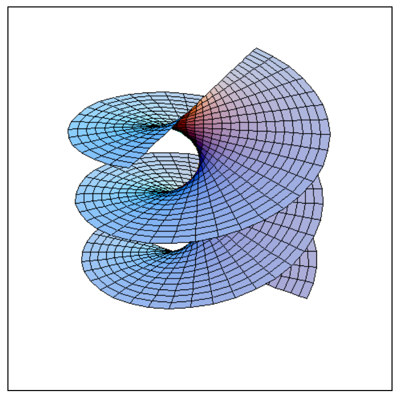
▷ **(a)** Write down a force $\mathbf{F}$ that has this property. Give arguments why your answer is correct.

Answer: Look at the clockwise path integral along a circle with radius R that goes around the origin according to $r = R$ and $\phi = -2\pi s$ as s goes from 0 to 1, that is: $\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 -F_\phi(R,\phi) \cdot 2\pi R \,\mathrm{d}s$. So for the force $\mathbf{F} = -c/(2\pi r)\hat{e}_\phi$, we have indeed the required property if the paths are such circles around O. Moreover, because this $\mathbf{F}$ does not have any $\hat{e}_r$ component, the same result holds if the path also includes movements in the $\hat{e}_r$ direction. Hence this $\mathbf{F}(r,\phi) = -c/(2\pi r)\hat{e}_\phi$ has the desired property.

▷ **(b)** Locally, in small patches that does not involve the origin, this force is conservative, and we can indeed give a local potential like function $V(r,\phi)$ with all the right properties. Yet globally no such potential should exist. What is going on here?

Answer: Indeed a 'quasi potential' like $V(r,\phi) = c\phi/2\pi$ gives $-\nabla V = -c/(2\pi r)\hat{e}_\phi$, which equals the force of the previous answer.

Note however that this function is not properly defined as it gives different values for $\phi = 0, 2\pi, 4\pi$ and so on. A drawing of such a potential would produce a *helicoid* (see figure), illustrating how by going around the origin one ends up c higher or lower than when started. If one requires that the potential function V is uniquely defined everywhere on the plane (say by restricting its definition to $0 \le \phi < 2\pi$), then it ends up being discontinuous (along $\phi = 0$).



Question 2. Time of impact under inverse quadratic force

We drop a particle with mass m at distance $r = d$ from the origin under the influence of a central potential $U(r) = -km/r$. Let s be the time required for the particle to reach the origin $r = 0$. As a function of m and d, it holds that $s = \gamma m^\alpha d^\beta$.

▷ **(a)** Determine these powers α and β.

Answer: This is a central force, hence conservation of energy must hold; at any given time t we must have position r and velocity v such that $E = -km/r + mv^2/2 = -km/d$. Rewriting this gives $v = -\sqrt{\frac{2k}{r} - \frac{2k}{d}}$ (the minus sign should be obvious), hence seperation of variables gives

$$\mathrm{d}t = -\frac{\mathrm{d}r}{\sqrt{\frac{2k}{r} - \frac{2k}{d}}}.$$

Integrating over the path from d to 0 (in time $t = 0$ to $t = s$) gives

$$s = -\frac{1}{\sqrt{2k}} \int_{r=d}^{r=0} \frac{\mathrm{d}r}{\sqrt{\frac{1}{r} - \frac{1}{d}}}$$

At this point it is already clear that $\alpha = 0$ (s does not depend on m), and because we are not interested in γ, we can ignore the term in front of the integral. Defining a new variable $z = r/d$ (with $\mathrm{d}r = d \cdot \mathrm{d}z$) turns this integral into

$$\int_{z=1}^{z=0} \frac{d \cdot \mathrm{d}z}{\sqrt{\frac{1}{zd} - \frac{1}{d}}} = d^{3/2} \int_{z=1}^{z=0} \frac{\mathrm{d}z}{\sqrt{\frac{1}{z} - 1}}, \tag{1}$$

hence $\beta = 3/2$.

Question 3. Orbits and Central Forces

A particle with mass m moves in the plane under influence of a central force $f(r)\hat{e}_r$. The trajectory of the particle is described by $r(t) = r_0 \mathrm{e}^{k \cdot \phi(t)}$ where $\phi(t)$ is the time dependent angle in the polar coordinate system that we are using.

▷ **(a)** Prove that $\phi(t)$ has to change logarithmically in time t.

Answer: Newton's $\mathbf{F} = m\mathbf{a}$ in polar coordinates tells us that $F_r = m(\ddot{r} - r\dot{\phi}^2)$ and $F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$, hence for central forces we have $r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$. For above mentioned $\mathbf{r}$ we have $\dot{r} = rk\dot{\phi}$, giving us $r\ddot{\phi} + 2rk\dot{\phi}^2 = 0$. The proposed $\phi = \gamma \log t$ is indeed a solution, as $r\ddot{\phi} + 2rk\dot{\phi}^2 = -r\gamma/t^2 + 2rk(\gamma/t)^2 = 0$ for $\gamma = 1/2k$. (This answer is of course not well defined for $t = 0$, but that can be resolved by shifting the time, and letting everything 'start' at $t = 1$. The general solution to the differential equation $r\ddot{\phi} + 2rk\dot{\phi}^2 = 0$ is given by $\phi(t) = \phi(0) + 1/2k \cdot \log(1 + Ct)$.)

▷ **(b)** Prove that $f(r)$ has to depend in an inverse cube way on r.

Answer: Here we use the relation given by the F_r component: $f(r)/m = \ddot{r} - r\dot{\phi}^2$. Using the answer $\phi(t) = (\log t)/2k$ to Question 3a, we have $r(t) = r_0 \mathrm{e}^{k\phi(t)} = r_0\sqrt{t}$, and hence $f(r)/m = -r_0/4 \cdot t^{-3/2} - r_0\sqrt{t}\gamma^2/t^2 = r_0(-1/4 - \gamma^2)t^{-3/2} \propto r^{-3}$.