

Classical Mechanics

Phys105A, Winter 2007

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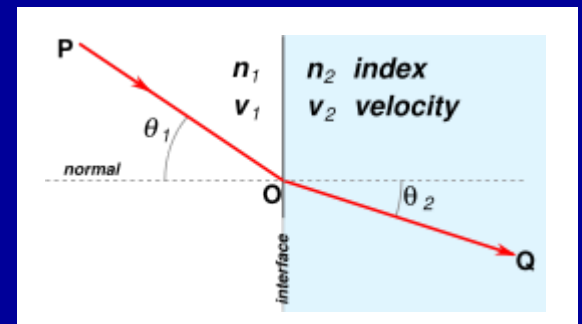
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Chapter 6: Calculus of Variations

Many calculations in physics can be rephrased as minimization problems.

Fermat's principle: When going from point P to Q, light takes that path that takes the least time.



“Given two media with different refractive indices, what is that path?”

More Minimization

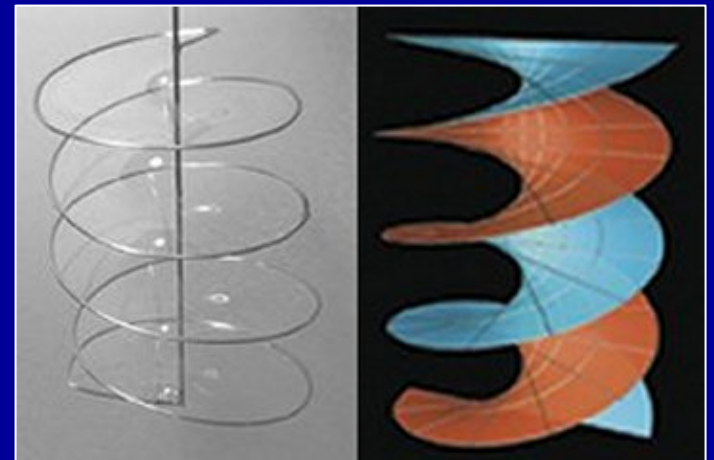
The natural, stable configuration of a necklace is that one that minimizes the potential energy of its chain.

“What is that shape?”



The stable shape of soap films is the one that minimizes its surface.

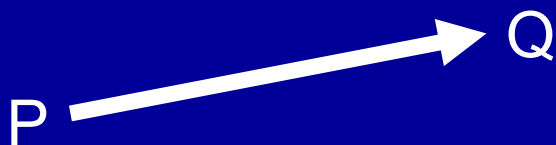
“For given boundary shapes, what is that soap shape?”



Shortest/Fastest Path

Given two points P, Q in \mathbb{R}^2 , what is the shortest path $y(x)$?

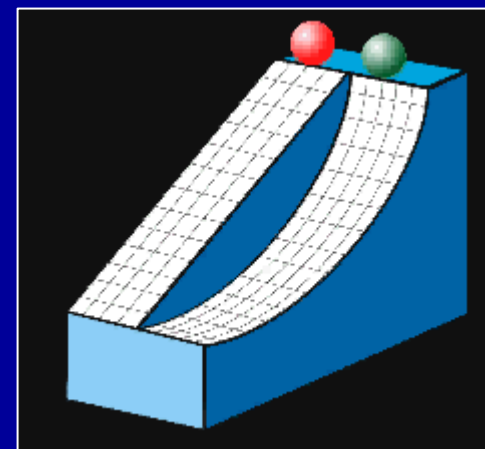
The path $y(x)$ should minimize the integral: $\int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$



Given two points P, Q in \mathbb{R}^2 in a gravitational field $g\mathbf{e}_x$, what is the fastest path $y(x)$?

With the velocity $v = \sqrt{2gx}$, the path $y(x)$ should minimize the integral:

$$\frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{x}} dx$$



Euler-Lagrange Equation

Let $y(x)$ be the path that minimizes/maximizes the integral

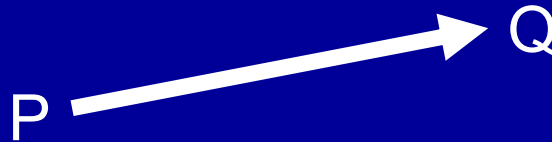
$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] dx$$

The Euler-Lagrange equation tells us that S is extremal when $y(x)$ obeys

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

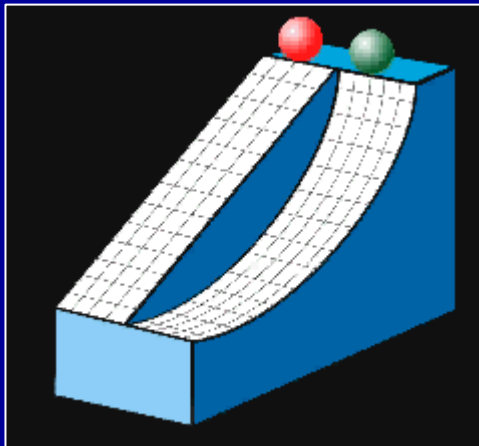
Shortest Path, Answer

Given two points P, Q in \mathbb{R}^2 , what is the shortest path $y(x)$?
The path $y(x)$ is a straight line ($y' = dy/dx = \text{constant}$).



Shortest/Fastest Path II

Given two points P, Q in \mathbb{R}^2 in a gravitational field $g\mathbf{e}_x$, what is the fastest path $y(x)$?



The answer to this classic *brachistochrone problem* is that $y(x)$ is (part) of a cycloid:

