Classical Mechanics

Phys105A, Winter 2007

Wim van Dam
Room 5109, Harold Frank Hall
vandam@cs.ucsb.edu
http://www.cs.ucsb.edu/~vandam/

Midterm

- New homework has been announced last Friday.
- The questions are the same as the Midterm
- It is due *this Friday.*

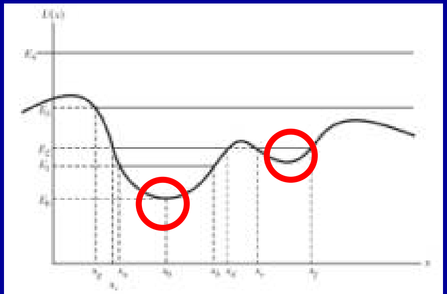
- Regarding the Midterm:
  Future homework assignments will be more aligned with the kind of questions for the Final.
- Suggestions, as always, are welcome.

Chapter 5: Oscillations

Hooke's Law

For a spring with *force constant* k (with units kg m/s^2) Hooke's Law states $F(x) = -kx$, such that the potential is $U(x) = ½kx^2$ (the system is stable as long as $k>0$).

All conservative, 1d, stable systems at x=0, can be approximated for small displacements x by such a parabolic U.



In other words: 1d, oscillating, conservative systems can always be approximated by Hooke's law (provided the oscillations are small enough).

Simple Harmonic Motion

The equation of motion is $d^2x/dt^2 = -(k/m)x = -\omega^2 x$ with the *angular frequency* $\omega=\sqrt{(k/m)}$. The general solution is the *superposition* $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$, which has *period* $\tau = 2\pi/\omega = 2\pi\sqrt{(m/k)}$ (with units s).

The constants C_1 and C_2 are determined by the position and velocity at (say) t=0.

We know of course that $e^{i\omega t} = \cos \omega t + \sin \omega t \sqrt{-1}$, yet x(t) will typically be real valued.
Hence the constants C_1 and C_2 will be such that the complex components 'cancel' each other.

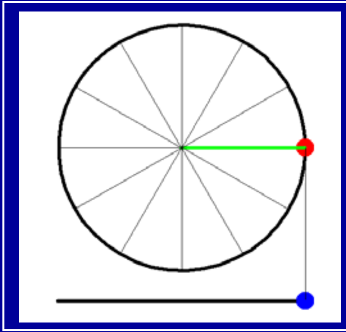
Solving the SHM

Equivalently, we can say we have the simple harmonic motion (SHM): $x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$,
where the requirement $x \in \mathbb{R}$ equals $B_1, B_2 \in \mathbb{R}$.

For initial (t=0) postion x_0 and velocity v_0, we get
$x(t) = x_0 \cos(\omega t) + (v_0/\omega) \sin(\omega t)$.

For general B_1, B_2, there is a phase shift $\delta = \tan^{-1}(B_2/B_1)$ with

$$B_1 \cos(\omega t) + B_2 \sin(\omega t) = \sqrt{B_1^2 + B_2^2} \cos(\omega t - \delta)$$



Another way of visualizing all this is as the x-coordinate of a circular motion:

Energy 'Flow' of a SHM

From now on $A = \sqrt{B_1^2 + B_2^2}$

The potential energy fluctuates as

$U = ½kx^2 = ½kA^2 \cos^2(\omega t – \delta)$

The kinetic energy goes like

$T = ½k(dx/dt)^2 = ½kA^2 \sin^2(\omega t – \delta)$

Hence the total energy we have

$E = T+U = ½kA^2$.

Two Dimensional Oscillations

For *isotropic* harmonic oscillators with **F** = –k**r** we get the solution (picking t=0 appropriately): $x(t) = A_x \cos(\omega t)$ and $y(t) = A_y \cos(\omega t - \delta)$.

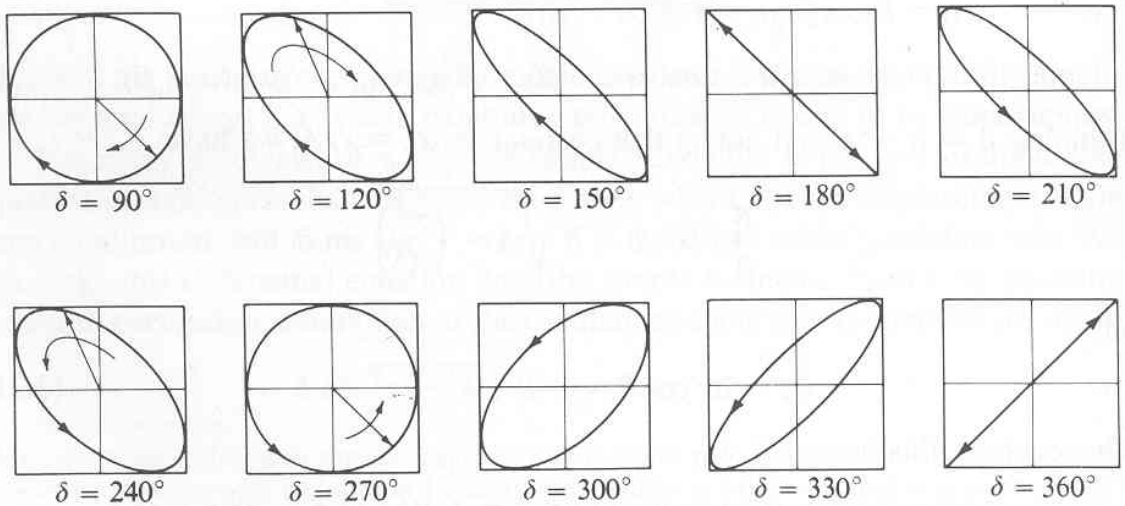


FIGURE 3-1

Anisotropic Oscillations

If (more generally) $F_x = k_x x$ and $F_y = k_y y$, then we have two independent oscillations, with solutions (again for right t=0): $x(t) = A_x \cos(\omega_x t)$ and $y(t) = A_y \cos(\omega_y t - \delta)$.

For such an anisotropic oscillator we have two angular frequencies $\omega_x = \sqrt{(k_x/m)}$ and $\omega_y = \sqrt{(k_y/m)}$.

Three cases when $\omega_x/\omega_y = ½$:

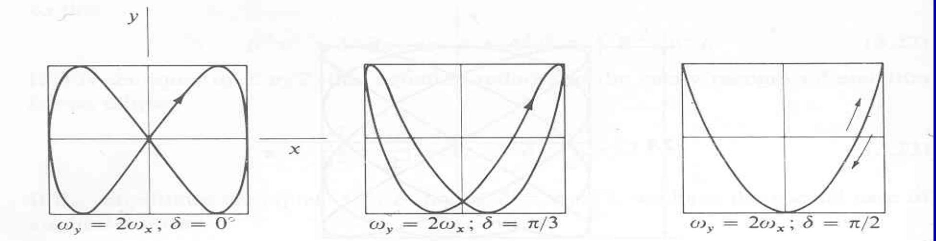


FIGURE 3-3

If ω_x/ω_y is irrational, the motion is *quasiperiodic* (see Taylor, page 172).

Damped Oscillations

Often an oscillating system will undergo a *resistive force* **f** = –b**v** that is linear in the velocity d**x**/dt (linear drag).

Thus, for a one dimensional, x-coordinate system, the combined force on the particle equals –kx –b dx/dt such that $md^2x/dt^2 = -kx - b\,dx/dt$, giving us the second order, linear, homogeneous differential equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

with m the mass of the particle, –bv the resistive force and –kx the Hooke's law force.

How to solve this damped oscillation?

Care versus Don't Care

We are mainly interested in the properties of the system that hold *regardless* of the initial conditions.

We care about: damping, frequencies,…

We care less about: specific velocities, angles, positions, and so on.

Differential Operators

Solving the equations of damped oscillations becomes significantly easier with the use of the differential operator $D = d/dt$, such that we can rewrite the equation as $mD^2x + bDx + kx = (mD^2 + bD + k)x = 0$, where D^2 stands for $D(D) = d^2/dt^2$.

To certain degree you can solve equations $f(D)x=0$ as if $f(D)$ is scalar valued: if $f(D)x=0$ and $g(D)x=0$, then we also have $\alpha f(D) g(D)x=0$ and $(\alpha f(D)+\beta g(D))x=0$.

An important exception occurs for $D^2x=0$: besides the solution $Dx=0$ (hence $x=c$), it can also refer to the case of x being linear ($x = at+c$) such that $Dx=a$, but $D^2x=0$.

Solving D Equations

With $D = d/dt$, take the differential equations $(D+4)x=0$.

Rewrite it as $Dx = -4x$

Observe that $x = C e^{-4t}$ is the general solution for $x(t)$.

Generally, $(D-a)x=0$ has the solution $x = C e^{at}$.

For 2nd order equations $f(D)x=0$ with $f(D)$ a quadratic polynomial in D, we solve the auxiliary equation $f(D)=0$ and use its solutions $D=a$ and $D=b$ to rewrite the equation as $(D-a)(D-b)x=0$. As a result, we have (typically) the solutions $x = C_1 e^{at}$ and $x = C_2 e^{bt}$.

If $a=b$, then $(D-a)^2x=0$ also gives: $x = C_2\, t\, e^{at}$.

Damped Oscillations

Often an oscillating system will undergo a *resistive force* $\mathbf{f} = -b\mathbf{v}$ that is linear in the velocity $d\mathbf{x}/dt$ (linear drag).

Thus, for a one dimensional, x-coordinate system, the combined force on the particle equals $-kx - b\,dx/dt$ such that $md^2x/dt^2 = -kx - b\,dx/dt$, giving us the second order, linear, homogeneous differential equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

with m the mass of the particle, $-bv$ the resistive force and $-kx$ the Hooke's law force.

How to solve this damped oscillation?

Solving D Equations, Take 2

With $D = d/dt$, $(D-a)x=0$ has the solution $x = C e^{at}$.

For a 2nd order equation $f(D)x=0$ with $f(D)$ a quadratic polynomial in D, we solve the *auxiliary equation* $f(D)=0$ and use its solutions $D=a$ and $D=b$ to rewrite the equation as $(D-a)(D-b)x=0$. As a result, we have (typically) the general solution $x = C_1 e^{at} + C_2 e^{bt}$.

If $a=b$, then $(D-a)^2x=0$ also gives: $x = C_2 t e^{at}$, giving the general solution $x = C_1 e^{at} + C_2 t e^{bt}$

What does this imply for the damped oscillation?

Solving the Equation

We rewrite the damped oscillation equation by defining $\beta=b/2m$ and $\omega_0=\sqrt{(k/m)}$ (both with frequency units 1/s) such that we have the equation $(D^2+2\beta D+\omega_0{}^2)x=0$.

Factorizing this gives:

$$(D+\beta+\sqrt{\beta^2-\omega_0^2})(D+\beta-\sqrt{\beta^2-\omega_0^2})x=0$$

There are thus three distinct scenarios:

$\beta<\omega_0$: "underdamping", when the drag –bv is small

$\beta>\omega_0$: "overdamping", when the drag –bv is large

$\beta=\omega_0$: "critical damping"

Weak Damping

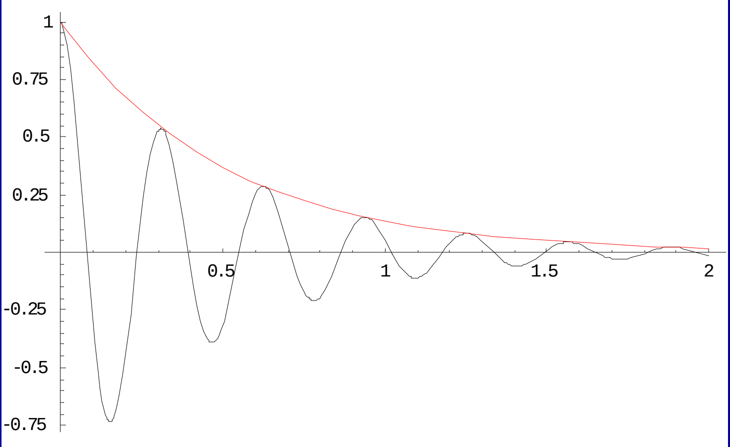
When $\beta<\omega_0$ we have for the differential equation

$$(D+\beta+\omega_1\sqrt{-1})(D+\beta-\omega_1\sqrt{-1})x=0$$

with $\omega_1=\sqrt{(\omega_0^2-\beta^2)}$, such that the general solution is

$$x(t)=e^{-\beta t}(C_1\cos\omega_1 t+C_2\sin\omega_1 t)$$
$$=A\cdot e^{-\beta t}\cdot\cos(\omega_1 t-\delta)$$

The decay factor is β, and the evolution looks like:



Note that for really small β we have $\omega_1\approx\omega_0$.

Strong Damping

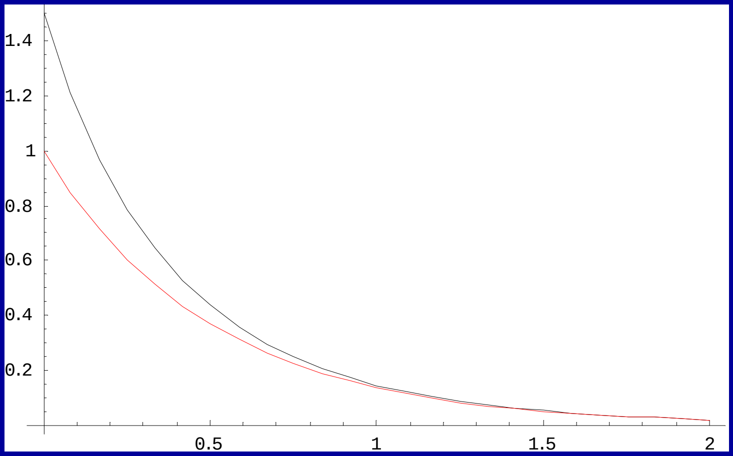
When $\beta > \omega_0$ we have for the differential equation

$$(D + \beta + \omega_1)(D + \beta - \omega_1)x = 0$$

such that the general solution is the sum of two decays:

$$x(t) = C_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}$$

The dominant decay factor is $\beta - \sqrt{(\beta^2 - \omega_0^2)}$,
and the evolution looks like:



Note that large β gives small decay factors.

Critical Damping

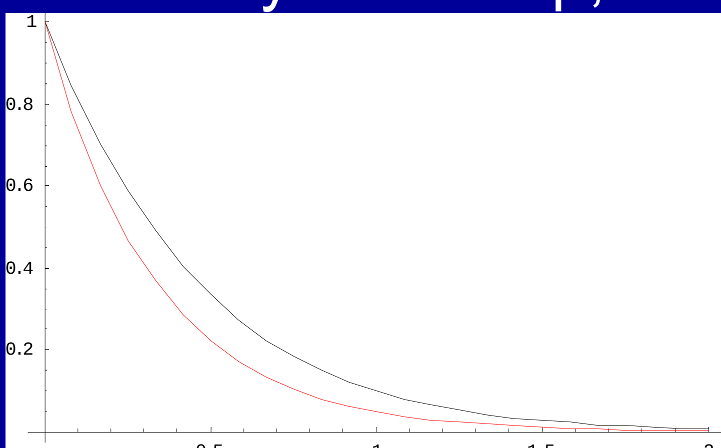
When $\beta=\omega_0$ we have for the differential equation

$$(D+\beta)(D+\beta)x=0$$

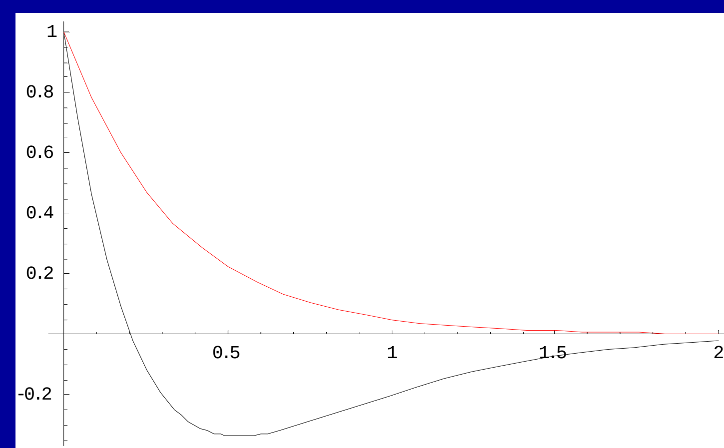
This time, the general solution is

$$x(t)=C_1e^{-\beta t}+C_2te^{-\beta t}$$

The decay factor is β, and the evolution looks like:



or



Driven Damped Oscillations

A damped oscillator (with m,b,k) driven by a time dependent force F(t) is described by the equation

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Rewriting with $2\beta = b/m$, $\omega_0 = \sqrt{(k/m)}$ and $f(t) = F(t)/m$ gives

$$(D^2 + 2\beta D + \omega_0^2)x = f(t)$$

This is an inhomogeneous differential equation, for which we know how to solve the homogeneous part.

We will describe a *particular solution* for $f = f_0 \cos \omega t$, where ω is the *driving frequency*.

Solving the Driven Oscillator

Solving the equation for the sinusoidal driving force

$$(D^2 + 2\beta D + \omega_0^2)x = f_0 \cos \omega t$$

gives…

$$x(t) = A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

With

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \qquad \delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

The C_1, C_2, r_1, r_2 are determined by the homogeneous equation and do not matter in the limit $t \to \infty$.