Artificial Intelligence

CS 165A

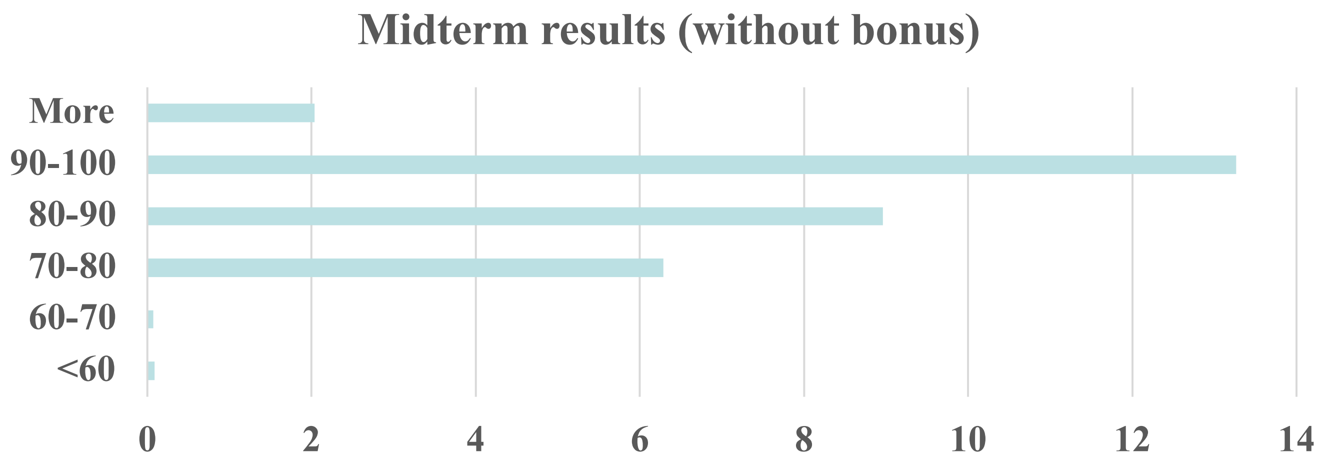
Nov 12, 2020

Instructor: Prof. Yu-Xiang Wang

Today

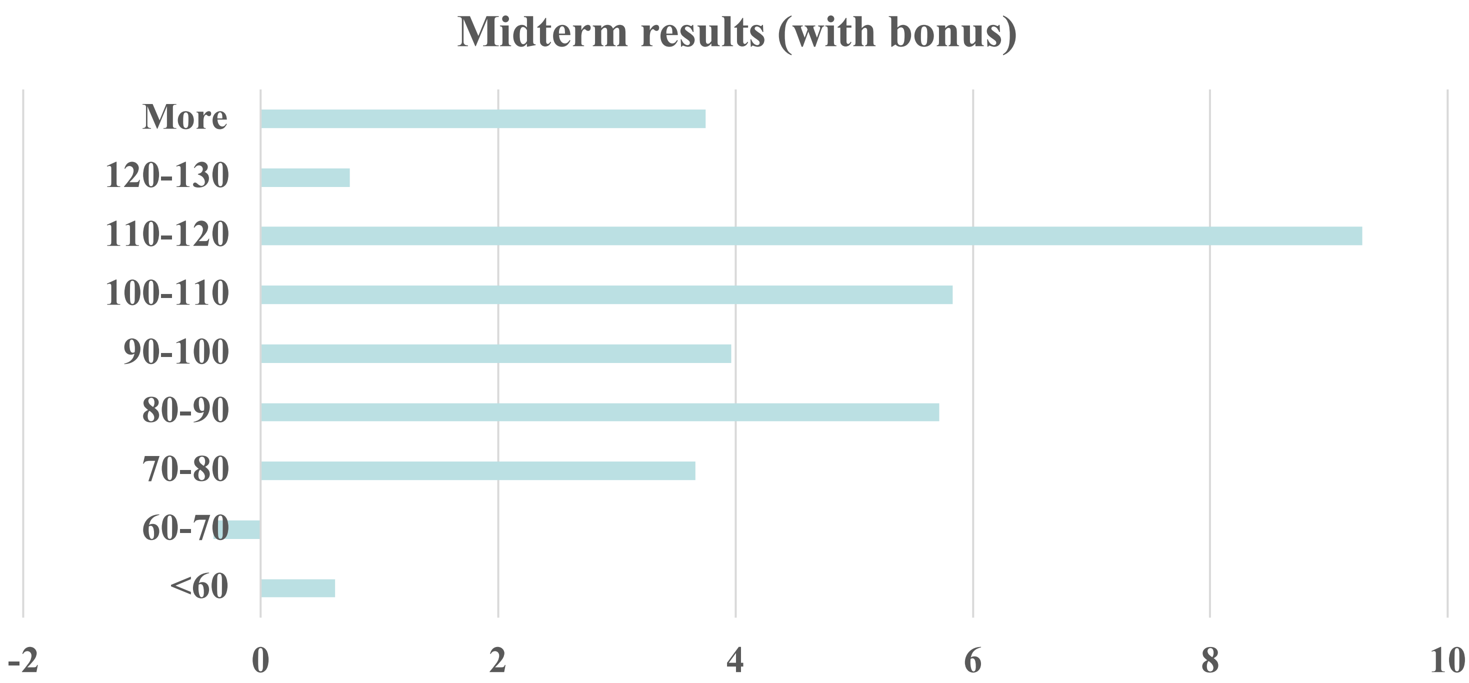
→ Markov Decision Processes

Midterm Results



(Histogram is sanitized using Differential Privacy)

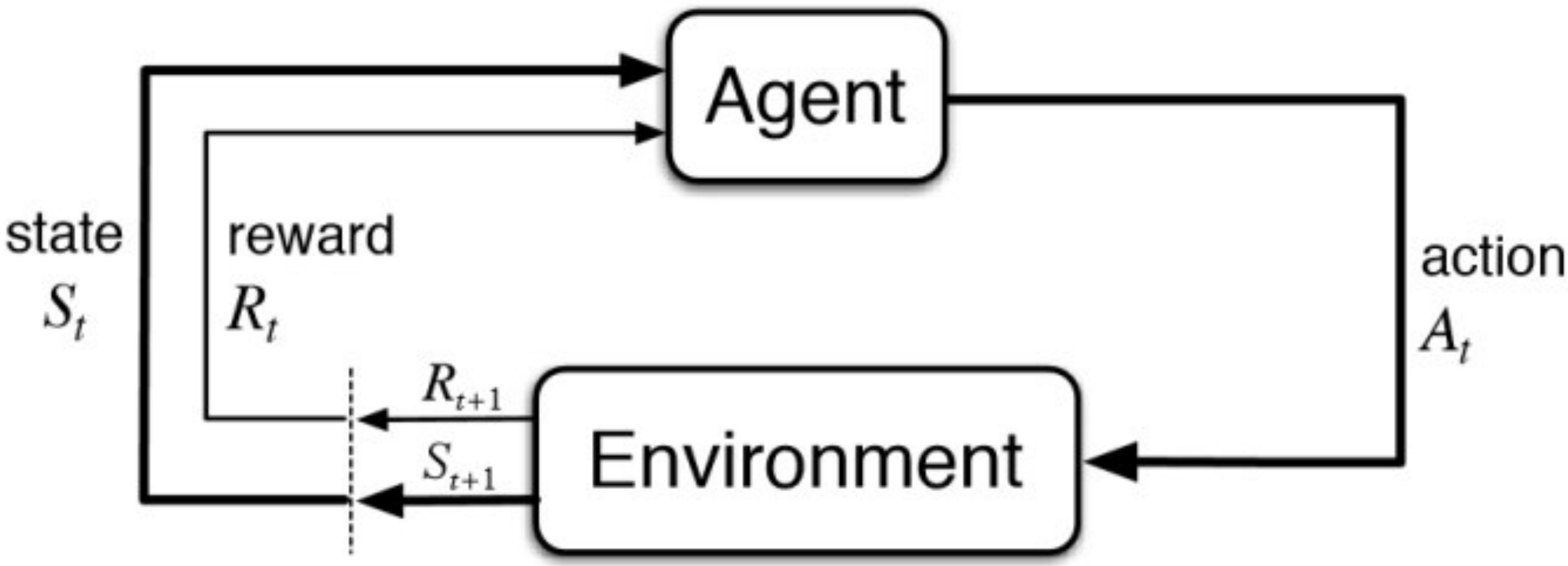
Midterm Results (with bonus)



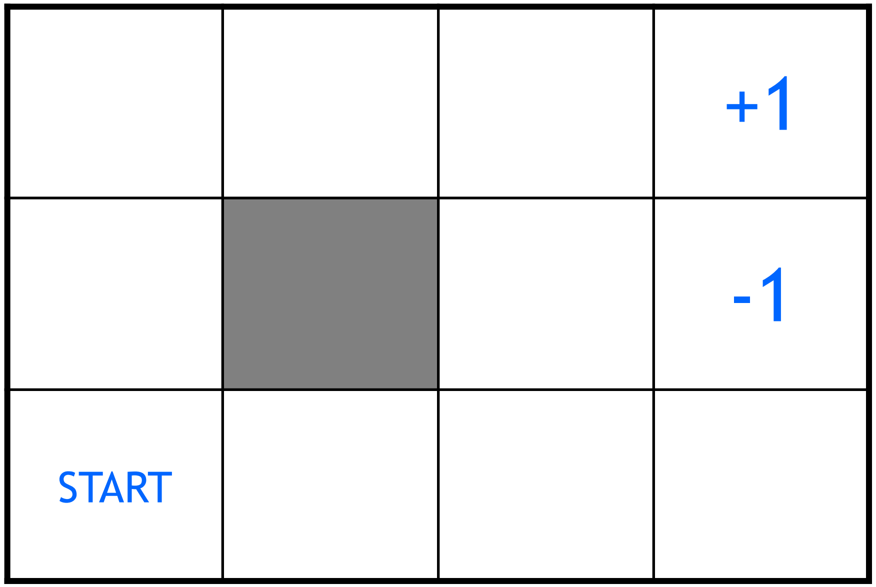
(Histogram is sanitized using Differential Privacy)

Recap: Reinforcement learning problem setup

- State, Action, Reward
- Unknown reward function, unknown state-transitions.
- Agents might not even observe the state



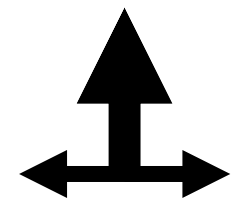
Recap: Robot in a room.



actions: UP, DOWN, LEFT, RIGHT

e.g.,

UP



State-transitions with action **UP**:

80%	move up
10%	move left
10%	move right

*If you bump into a wall, you stay where you are.

- reward +1 at [4,3], -1 at [4,2]
- reward -0.04 for each step
- what's the strategy to achieve max reward?
- what if the transitions were deterministic?

Recap: Tabular MDP

- Discrete State, Discrete Action, Reward and Observation

$$S_t \in \mathcal{S} \quad A_t \in \mathcal{A} \quad R_t \in \mathbb{R} \quad \cancel{O_t \in \mathcal{O}}$$

- Policy:
  - When the state is observable: $\pi : \mathcal{S} \rightarrow \mathcal{A}$
  - ~~Or when the state is not observable~~

$$\cancel{\pi_t : (\mathcal{O} \times \mathcal{A} \times \mathbb{R})^{t-1} \rightarrow \mathcal{A}}$$

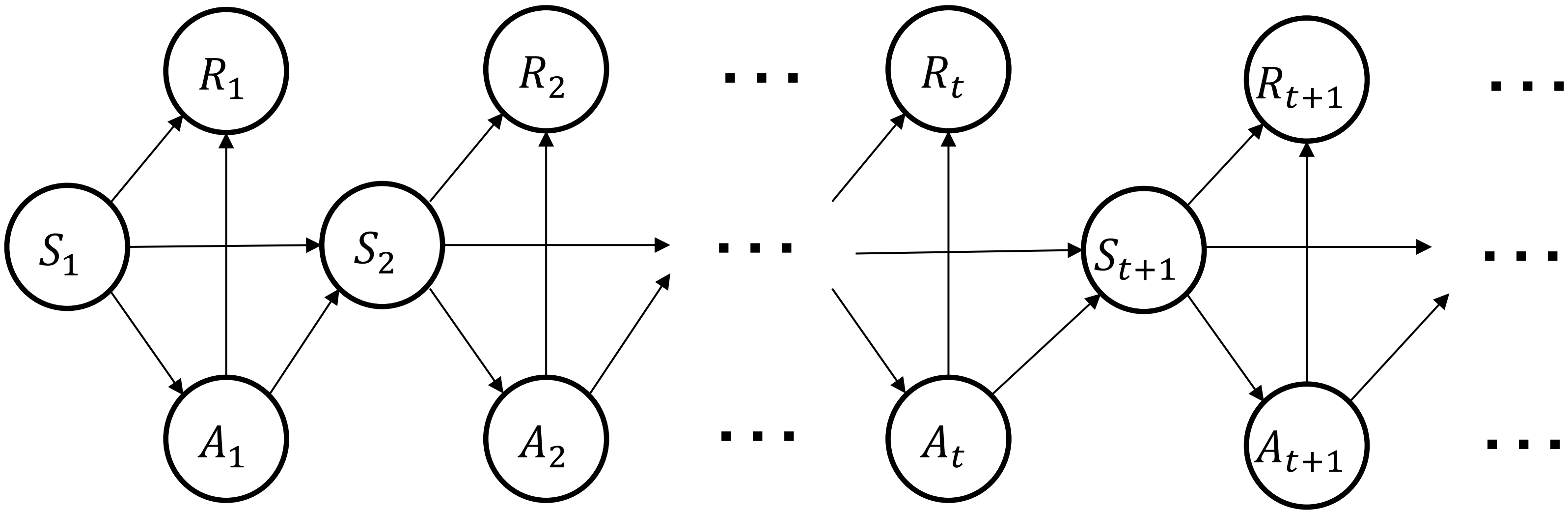
- Learn the best policy that maximizes the expected reward
  - Finite horizon (episodic) RL: $\pi^* = \arg\max_{\pi \in \Pi} \mathbb{E}[\sum_{t=1}^{T} R_t]$

T: horizon

  - Infinite horizon RL: $\pi^* = \arg\max_{\pi \in \Pi} \mathbb{E}[\sum_{t=1}^{\infty} \gamma^{t-1} R_t]$

γ: discount factor

Recap: Parameters of an MDP are the CPTs



- Initial state distribution
- Transition dynamics
- Reward distribution

Recap: Reward function and Value functions

- Immediate reward function r(s,a,s')
  - expected **immediate** reward

$$r(s,a,s') = \mathbb{E}[R_1 | S_1 = s, A_1 = a, S_2 = s']$$

$$r^{\pi}(s) = \mathbb{E}_{a \sim \pi(a|s)}[R_1 | S_1 = s]$$

- state value function: $V^{\pi}(s)$
  - expected **long-term** return when starting in *s* and following π

$$V^{\pi}(s) = \mathbb{E}_{\pi}[R_1 + \gamma R_2 + ... + \gamma^{t-1} R_t + ... | S_1 = s]$$

- state-action value function: $Q^{\pi}(s,a)$
  - expected **long-term** return when starting in *s*, performing *a,* and following π

$$Q^{\pi}(s,a) = \mathbb{E}_{\pi}[R_1 + \gamma R_2 + ... + \gamma^{t-1} R_t + ... | S_1 = s, A_1 = a]$$

Recap: Bellman equations – the fundamental equations of MDP and RL

- An alternative, recursive and more useful way of defining the V-function and Q function

$$V^\pi(s) = \sum_a \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^\pi(s')] = \sum_a \pi(a|s) Q^\pi(s,a)$$

- Quiz:
  - Prove Bellman equation from the definition in the previous slide.
  - Write down the Bellman equation using Q function alone.

    $Q^\pi(s,a) = ?$

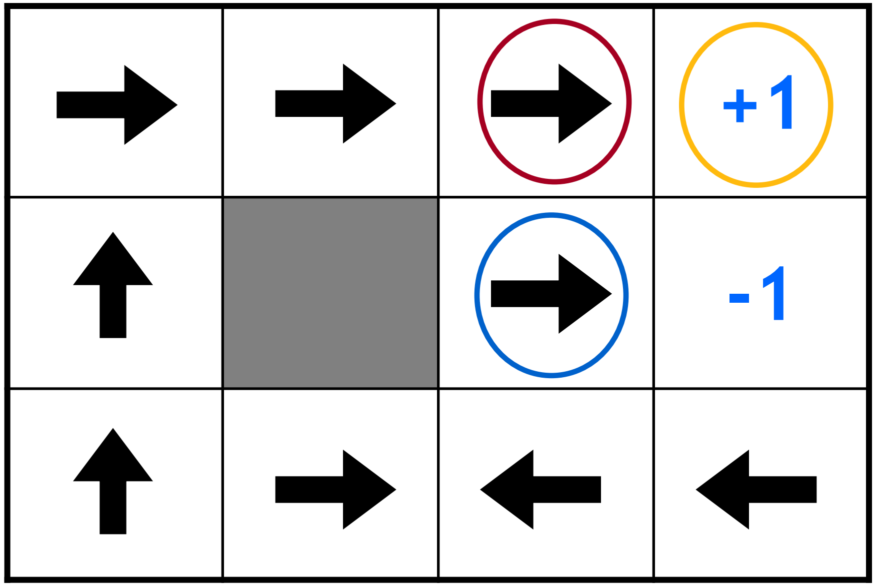
This lecture

- Bellman equations

- Algorithms for solving MDPs
  - Value iterations / Policy Iterations

- Exploration and Bandit problem

Let's work out the Value function for a specific policy



actions: UP, DOWN, LEFT, RIGHT

e.g., **UP**

state-transitions with action **UP**:

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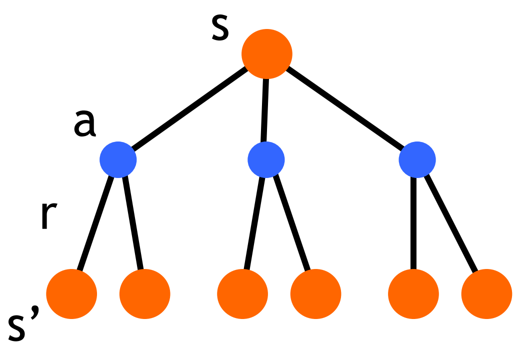
$$V^\pi(s) = \sum_a \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^\pi(s')] = \sum_a \pi(a|s) Q^\pi(s,a)$$

1.0   0.8 * (+1-0.04 + 0)

\+ 0.1 * (-0.04 + V^π([3,2]))

\+ 0.1 * (-0.04 + V^π([3,3]))

Optimal value functions



- there's a set of *optimal* policies
  - V^π defines partial ordering on policies
  - they share the same optimal value function

$$V^*(s) = \max_\pi V^\pi(s)$$

- Bellman optimality equation

$$V^*(s) = \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^*(s')]$$

  - system of n non-linear equations
  - solve for V*(s)
  - easy to extract the optimal policy

- having Q*(s,a) makes it even simpler

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

Inference problem: given an MDP, how to compute its optimal policy?

- It suffices to compute its Q* function, because:

$$\pi^*(s) = \arg\max_a Q^*(s, a)$$

- It suffices to compute its V* function, because:

$$Q^*(s, a) = \sum_{s'} P(s'|s, a)[r(s, a, s') + \gamma V^*(s')]$$

Summary of Bellman equations – the fundamental equations of MDP and RL

- V-function and Q function
  - V^π function Bellman equation

$$V^\pi(s) = \sum_a \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^\pi(s')]$$

  - Q^π function Bellman equation

$$Q^\pi(s,a) = \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \sum_{a'} \pi(a'|s') Q^\pi(s',a')]$$

  - V* function Bellman (optimality) equation

$$V^*(s) = \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^*(s')]$$

  - Q* function Bellman (optimality) equation

$$Q^*(s,a) = \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \max_{a'} Q^*(s',a')]$$

Algorithms for calculating the V* function

- Policy evaluation, policy-improvement

- Policy iterations

- Value iterations

Dynamic programming

- main idea
  - use value functions to structure the search for good policies
  - need a known model of the environment

- two main components
  - policy evaluation: compute V^π from π
  - policy improvement: improve π based on V^π

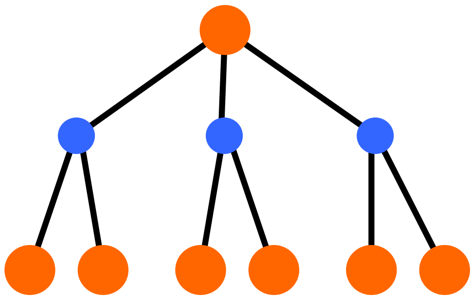
  - start with an arbitrary policy
  - repeat evaluation/improvement until convergence

Policy evaluation/improvement

- policy evaluation: π -> V^π
  - Bellman eqn's define a system of n eqn's
  - could solve, but will use iterative version

$$V^\pi_{k+1}(s) \leftarrow \sum_a \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^\pi_k(s')]$$

  - start with an arbitrary value function V_0, iterate until V_k converges



- policy improvement: V^π -> π'

$$\begin{aligned}\pi'(s) &= \arg\max_a Q^\pi(s,a) \\ &= \arg\max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^\pi_k(s')]\end{aligned}$$

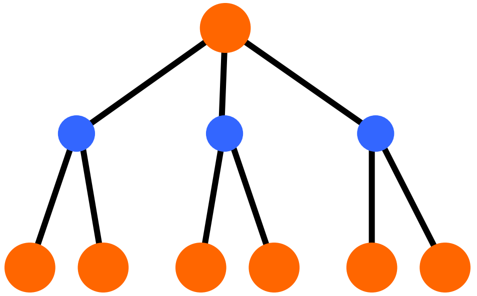
  - π' either strictly better than π, or π' is optimal (if π = π')

Policy/Value iteration

- Policy iteration

$$\pi_0 \rightarrow^E V^{\pi_0} \rightarrow^I \pi_1 \rightarrow^E V^{\pi_1} \rightarrow^I \ldots \rightarrow^I \pi^* \rightarrow^E V^*$$

  - two nested iterations; too slow
  - don't need to converge to V^{π_k}
    - just move towards it

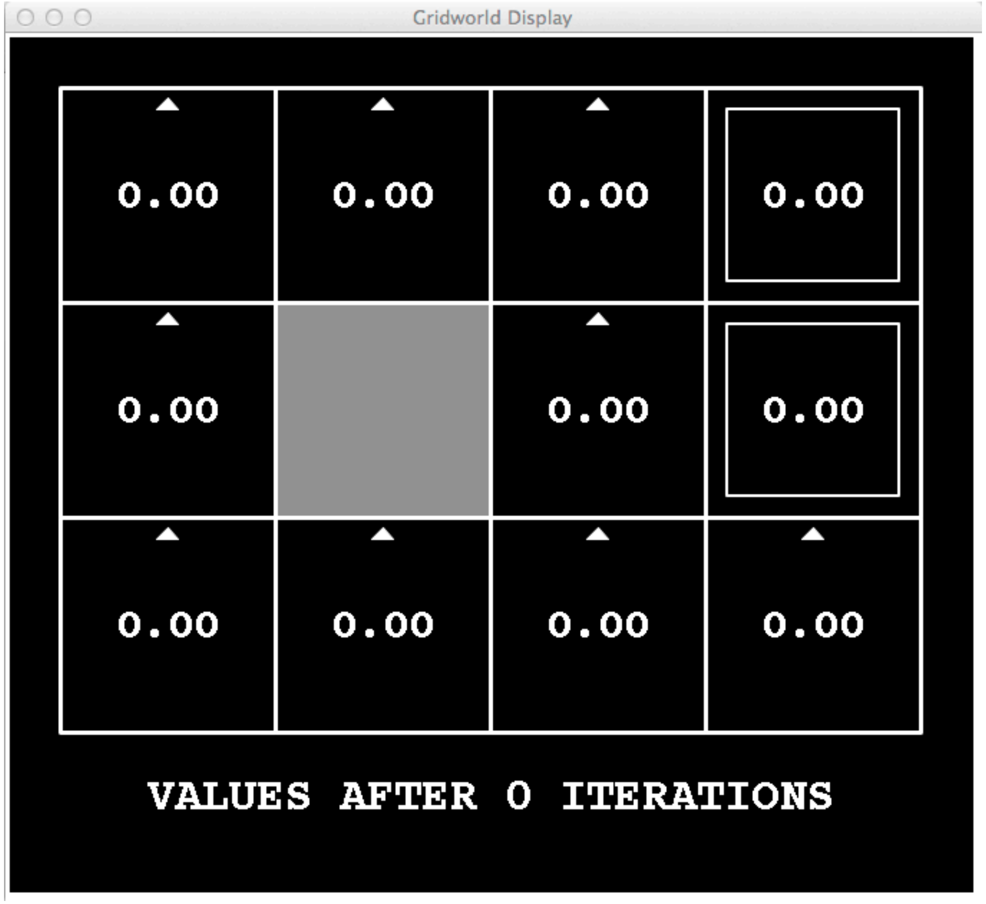


- Value iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k(s')]$$

  - use Bellman optimality equation as an update
  - converges to V*

k=0

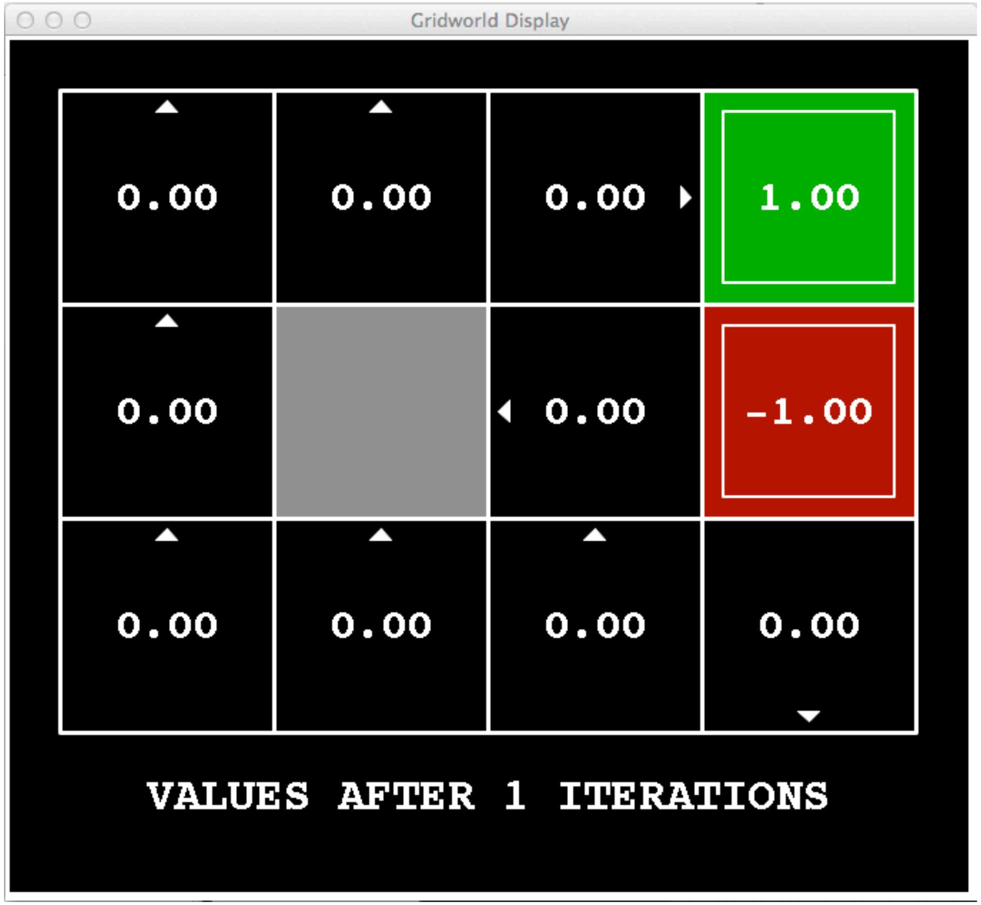


Noise = 0.2

Discount = 0.9

Living reward = 0

k=1



Noise = 0.2

Discount = 0.9

Living reward = 0

k=2

Noise = 0.2

Discount = 0.9

Living reward = 0

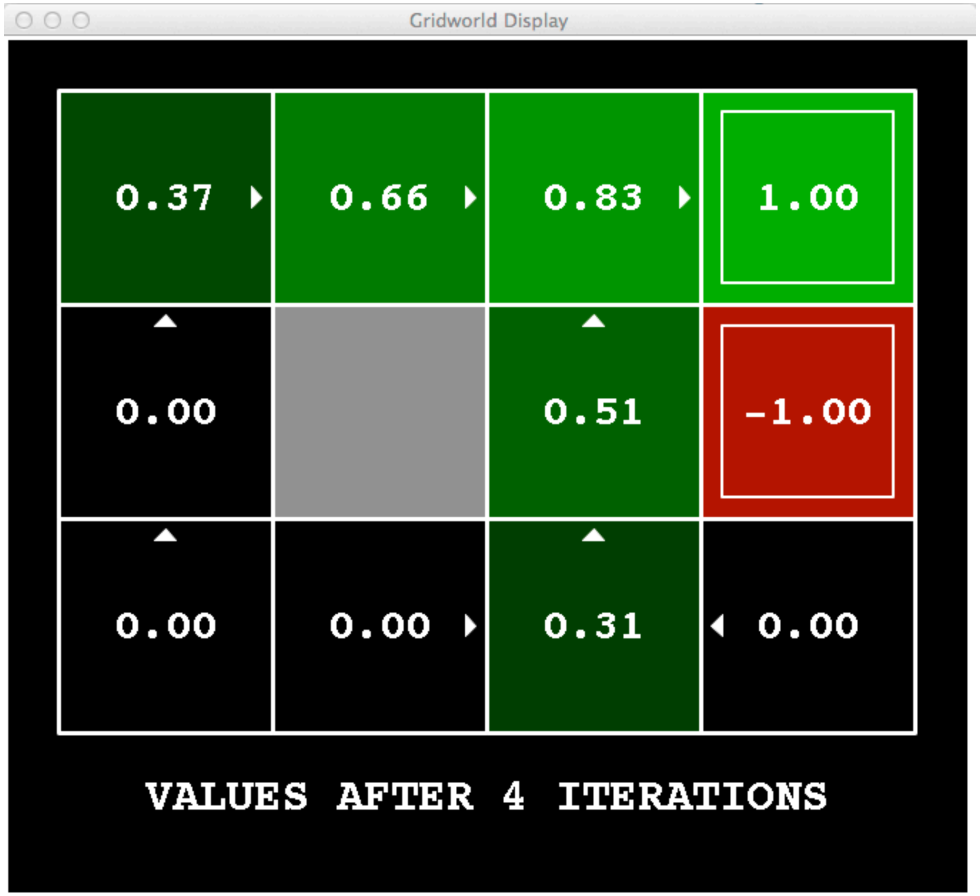
k=3

Noise = 0.2

Discount = 0.9

Living reward = 0

k=4

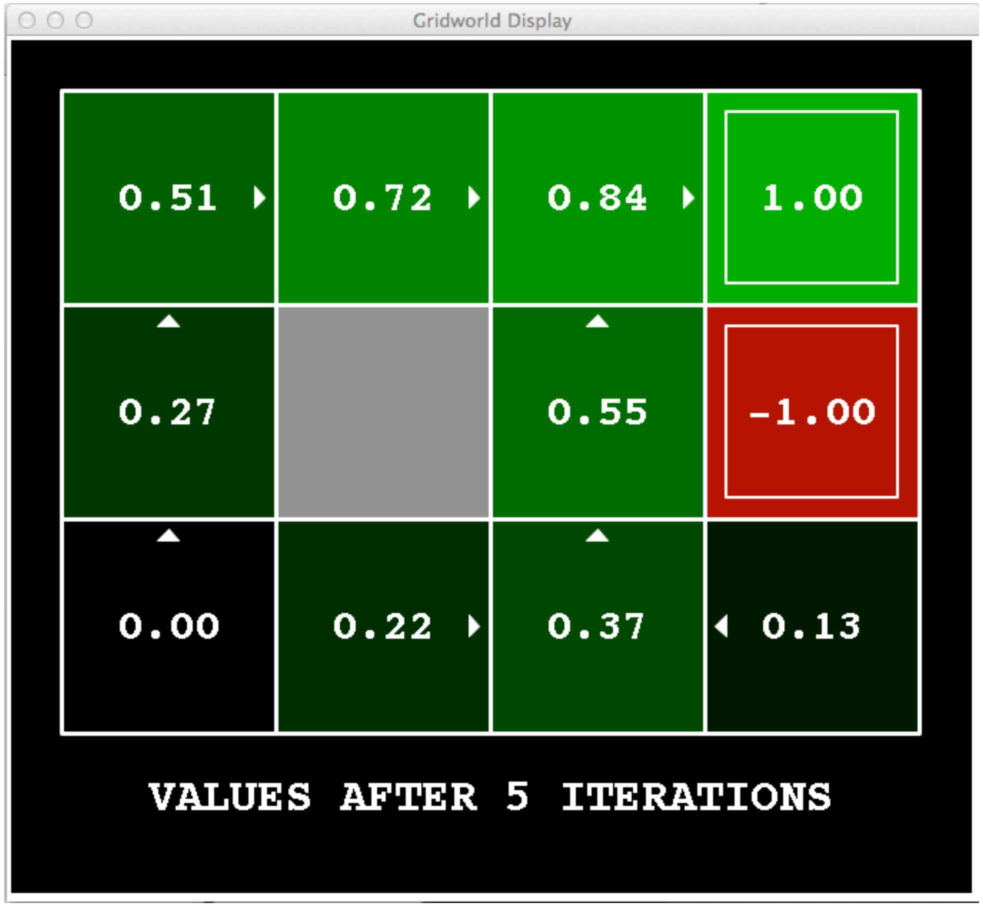


Noise = 0.2

Discount = 0.9

Living reward = 0

k=5

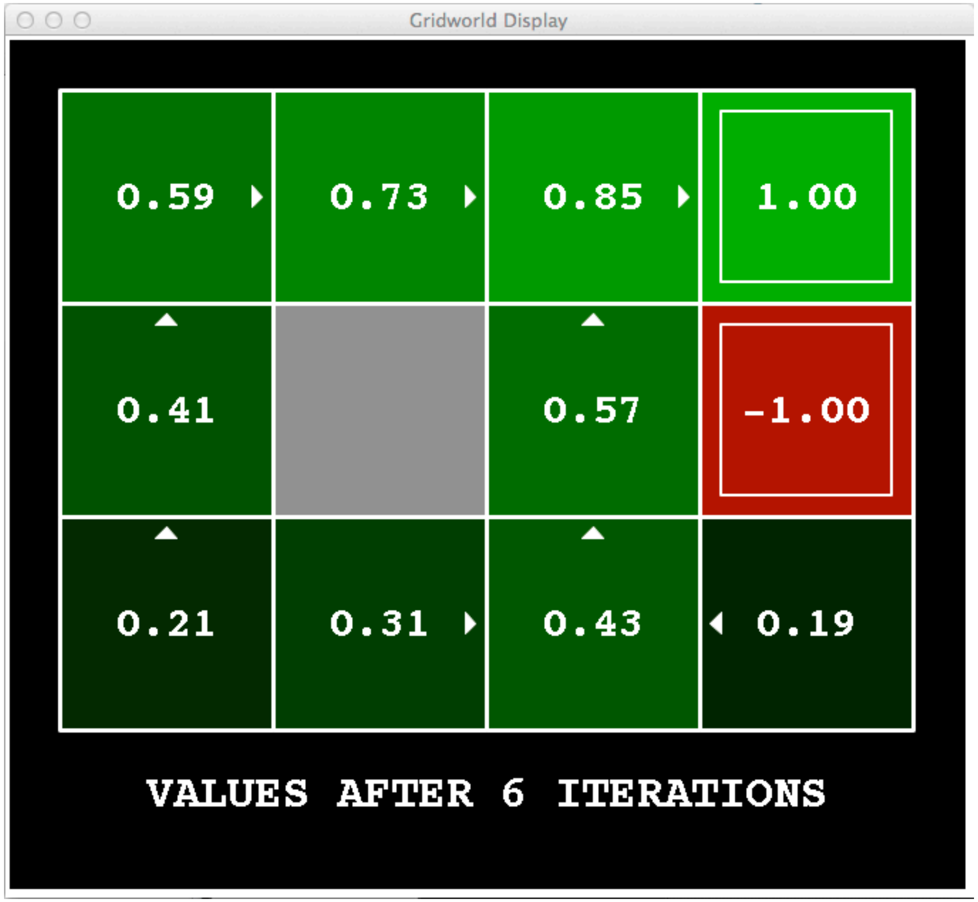


Noise = 0.2

Discount = 0.9

Living reward = 0

k=6

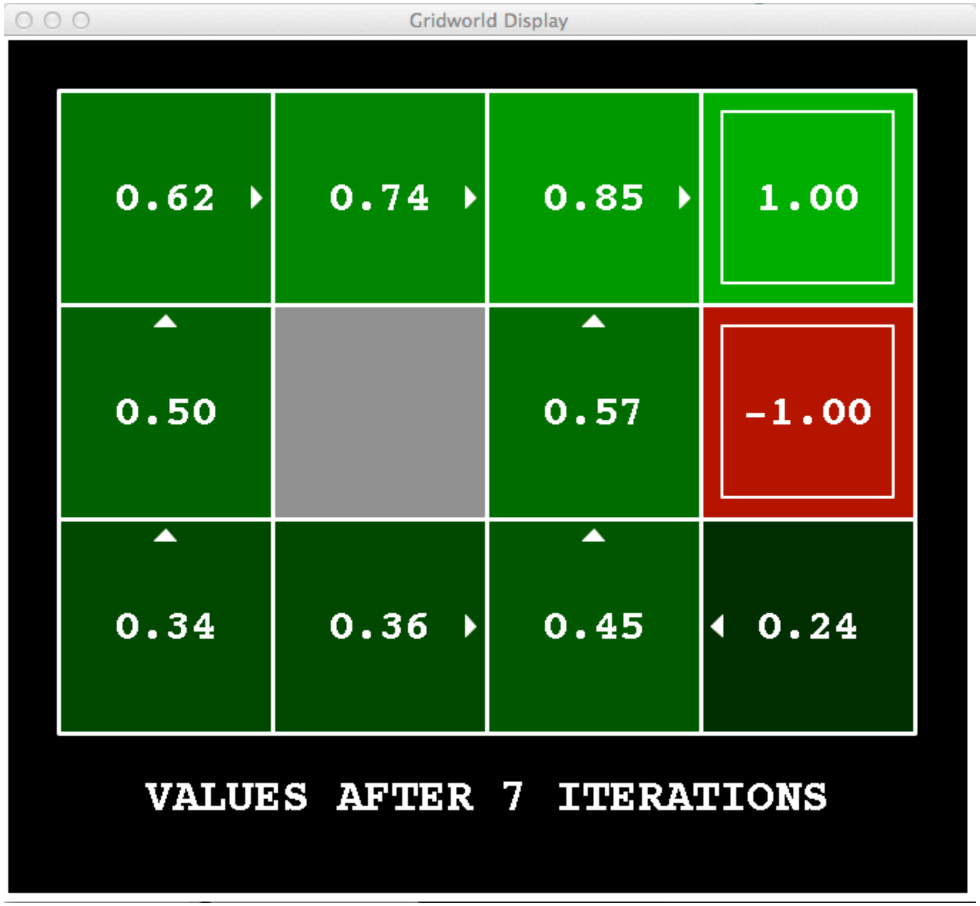


Noise = 0.2

Discount = 0.9

Living reward = 0

k=7

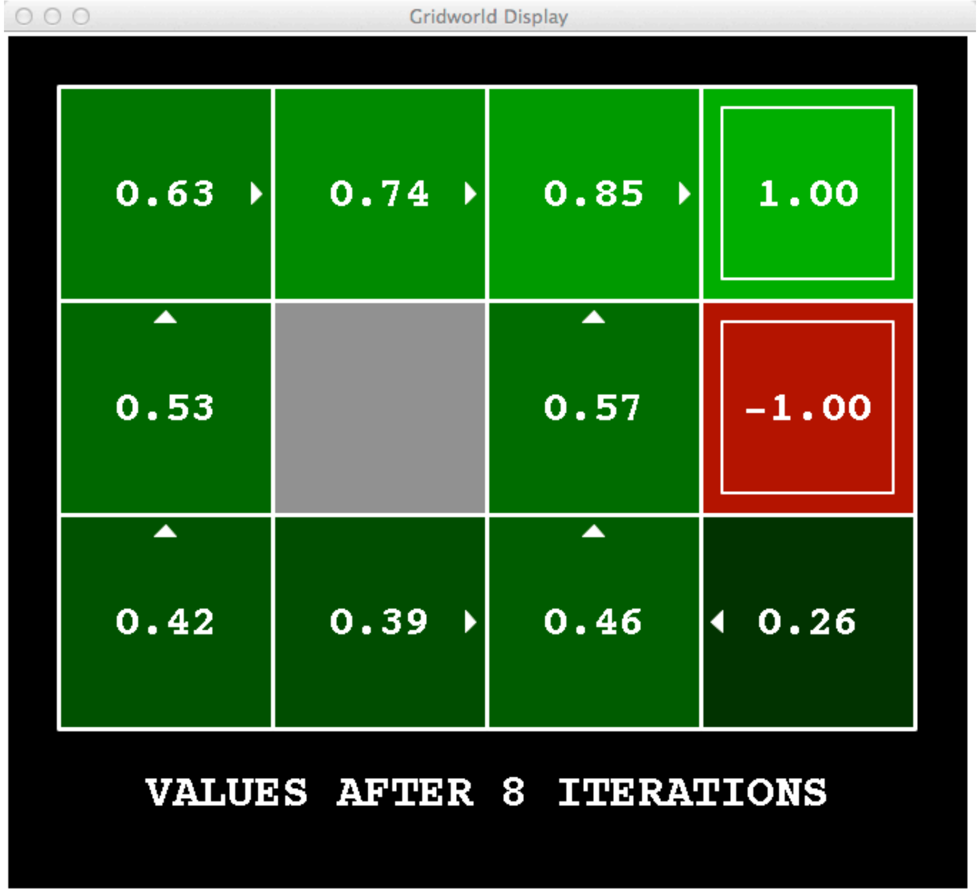


Noise = 0.2

Discount = 0.9

Living reward = 0

k=8

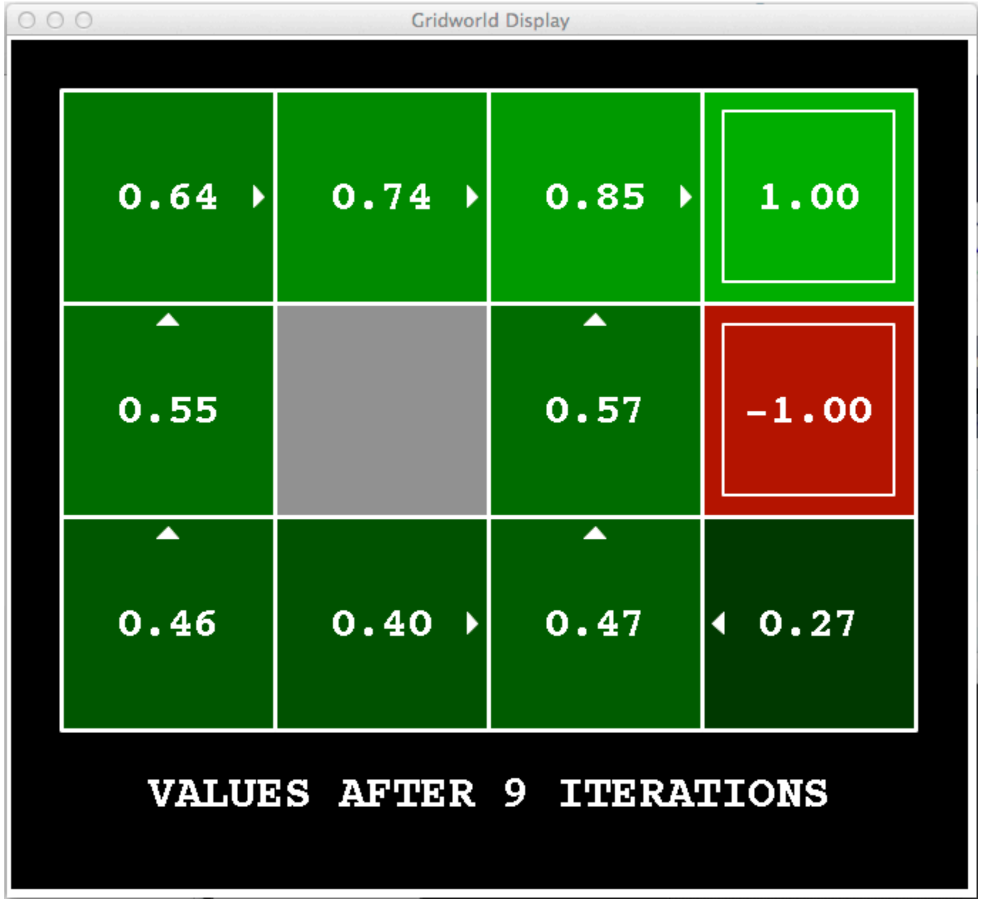


Noise = 0.2

Discount = 0.9

Living reward = 0

k=9

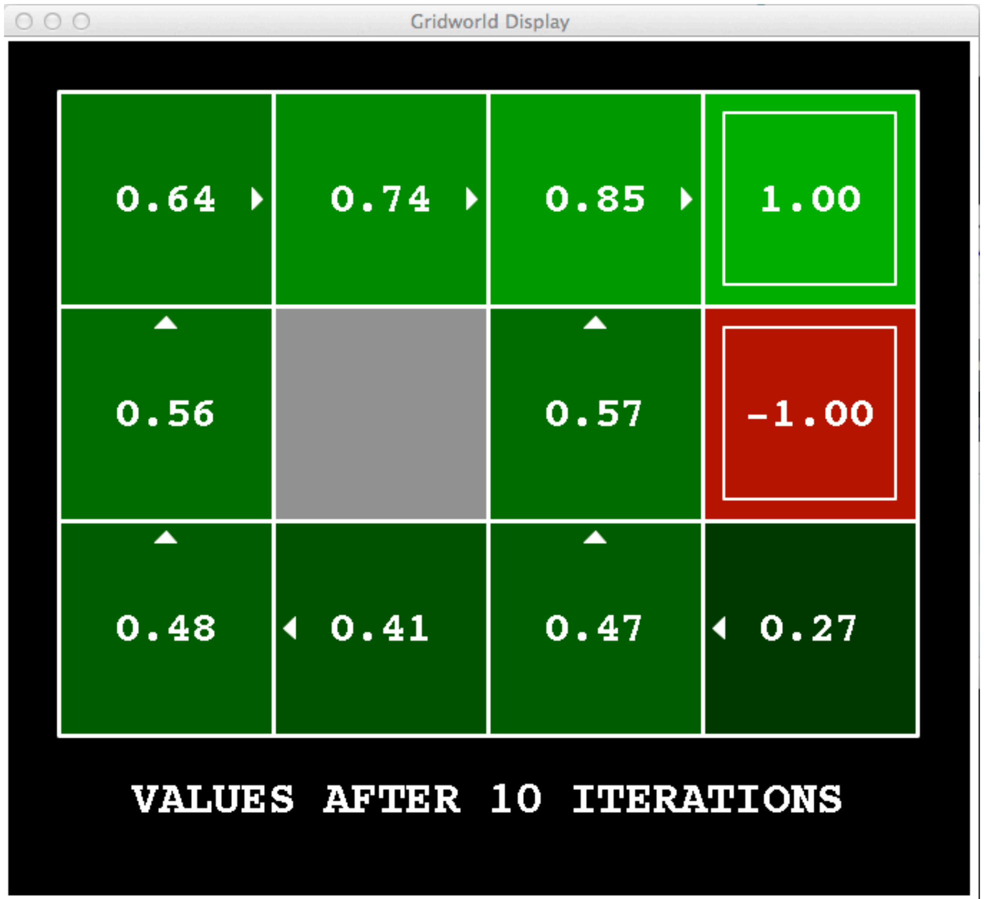


Noise = 0.2

Discount = 0.9

Living reward = 0

k=10

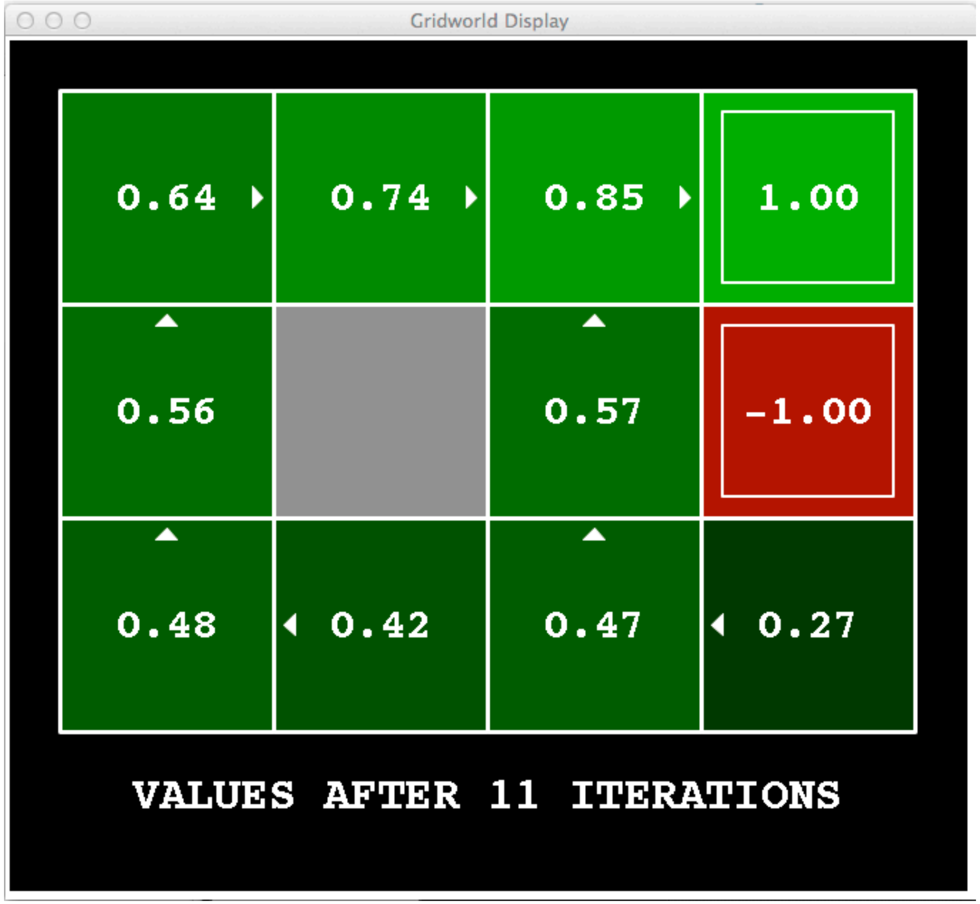


Noise = 0.2

Discount = 0.9

Living reward = 0

k=11

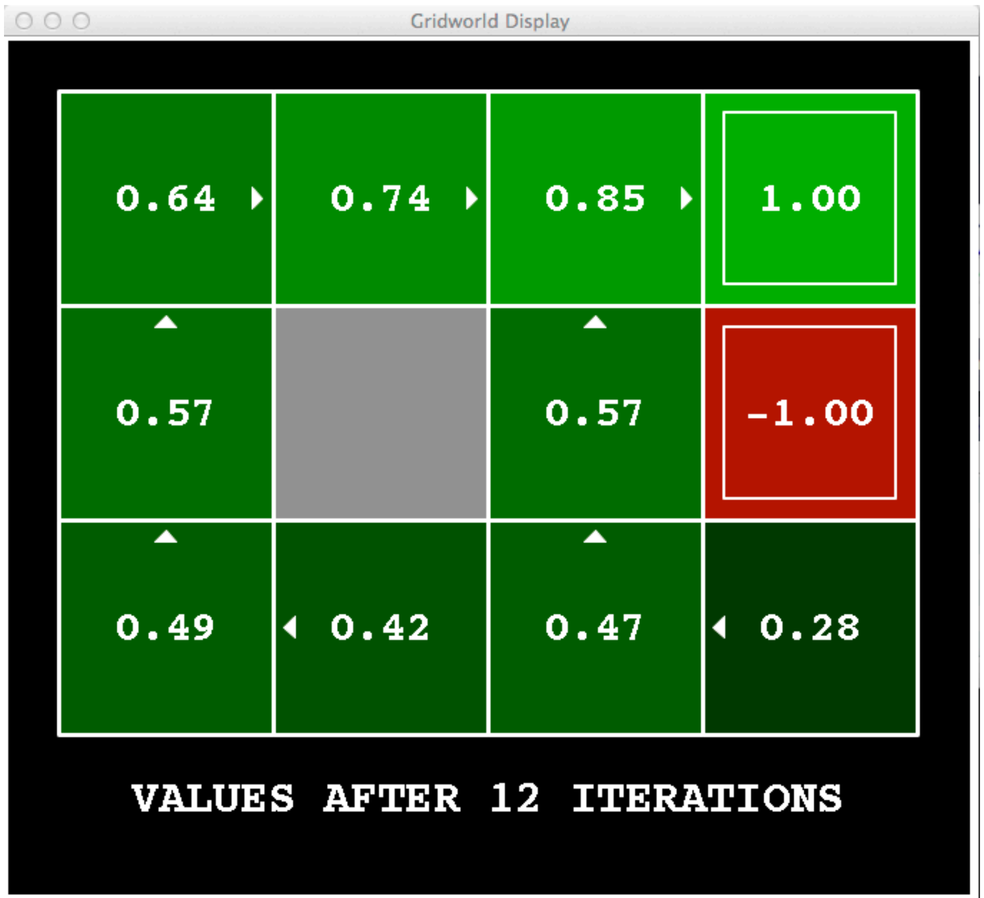


Noise = 0.2

Discount = 0.9

Living reward = 0

k=12

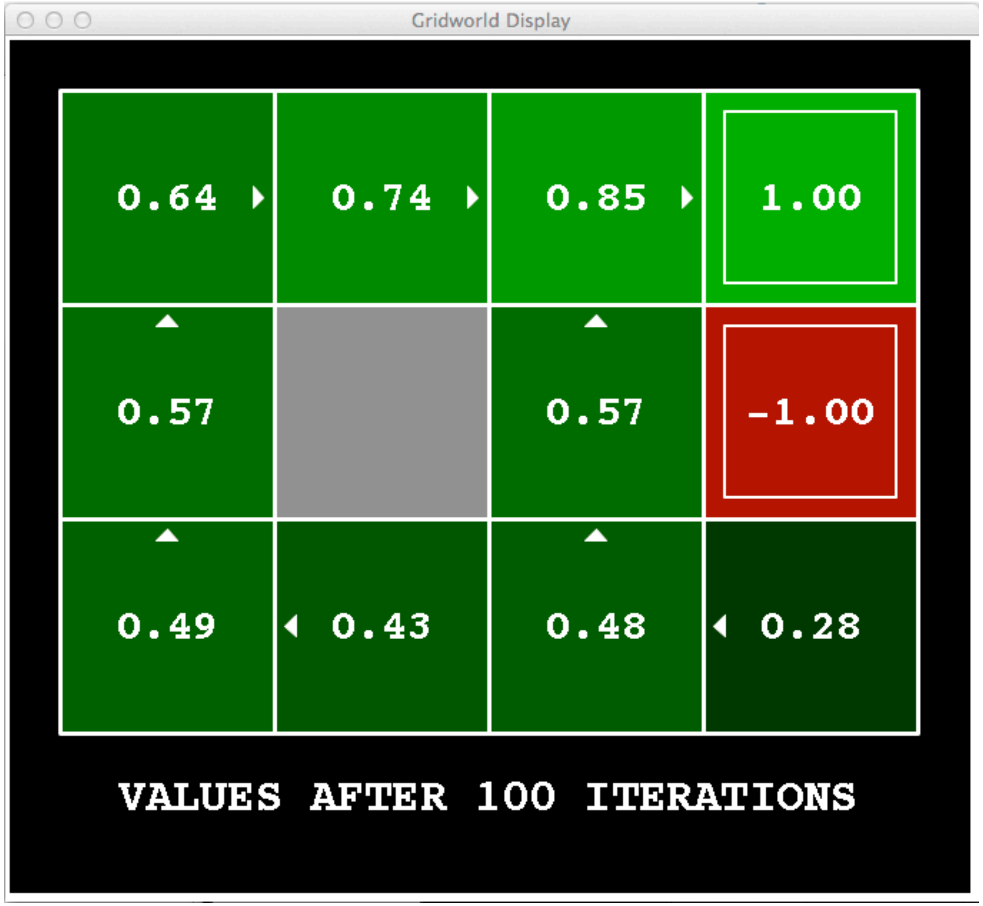


Noise = 0.2

Discount = 0.9

Living reward = 0

k=100



Noise = 0.2

Discount = 0.9

Living reward = 0

Q-iteration

- Updating Q functions instead of V functions

$$Q_{k+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \max_{a'} Q_k(s',a')]$$

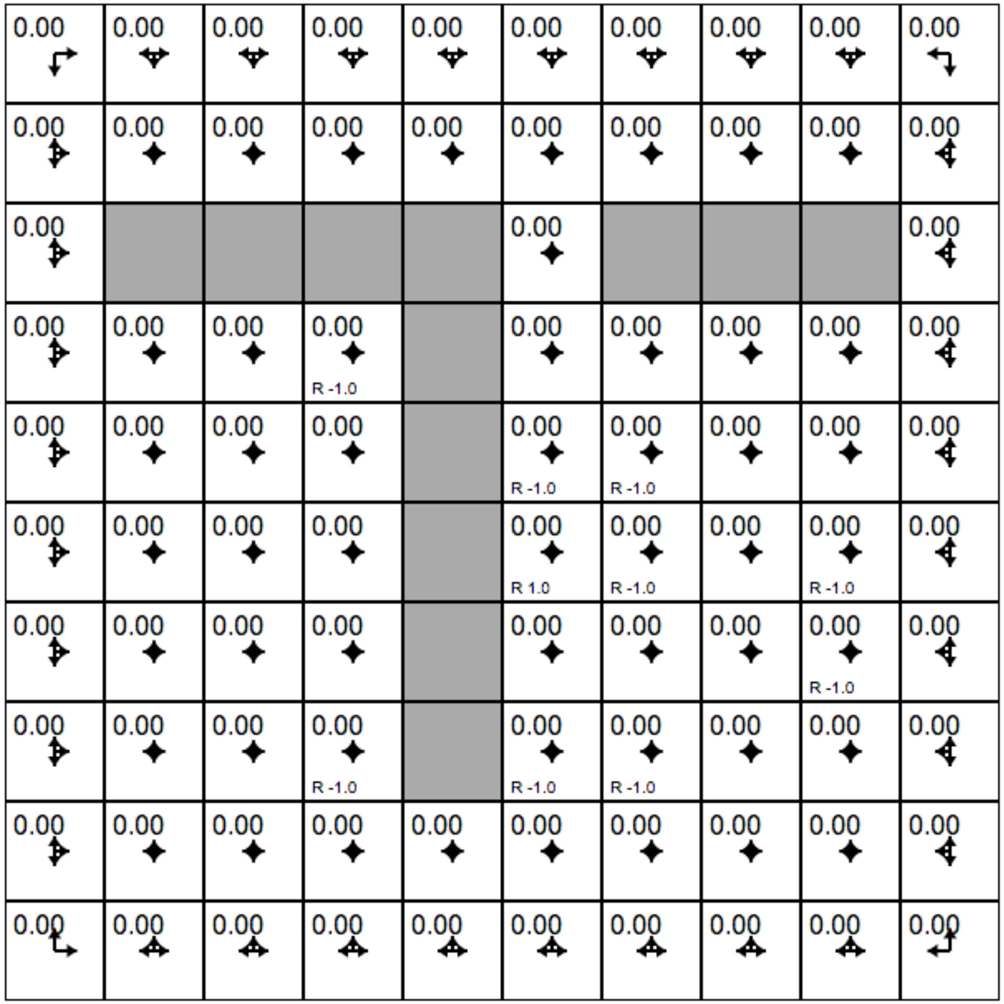
- Quiz: What is the difference from the following extended version of value iteration?

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k(s')]$$

$$Q_{k+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_{k+1}(s')]$$

Ans: They are identical!

Demo: grid worlds



https://cs.stanford.edu/people/karpathy/reinforcejs/gridworld_dp.html

MDP summary

- Tabular MDP
- Episodic vs. infinite horizon (discounted)
- Immediate reward vs long-term reward
- Value functions: V functions, Q functions
- Bellman equations, Bellman optimality equations
- How to solve MDP? Policy iterations, value iterations

MDP Summary

Standard expectimax: $V(s) = \max_a \sum_{s'} P(s'|s,a)V(s')$

Bellman equations: $V(s) = \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V(s')]$

Value iteration: $V_{k+1}(s) = \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k(s')], \quad \forall\, s$

Q-iteration: $Q_{k+1}(s,a) = \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \max_{a'} Q_k(s',a')], \quad \forall\, s,a$

Policy evaluation: $V^{\pi}_{k+1}(s) = \sum_{s'} P(s'|s,\pi(s))[r(s,\pi(s),s') + \gamma V^{\pi}_k(s')], \quad \forall\, s$

Policy improvement: $\pi_{new}(s) = \operatorname{argmax}_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^{\pi_{old}}(s')], \quad \forall\, s$

Matrix-form of Bellman Equations and VI

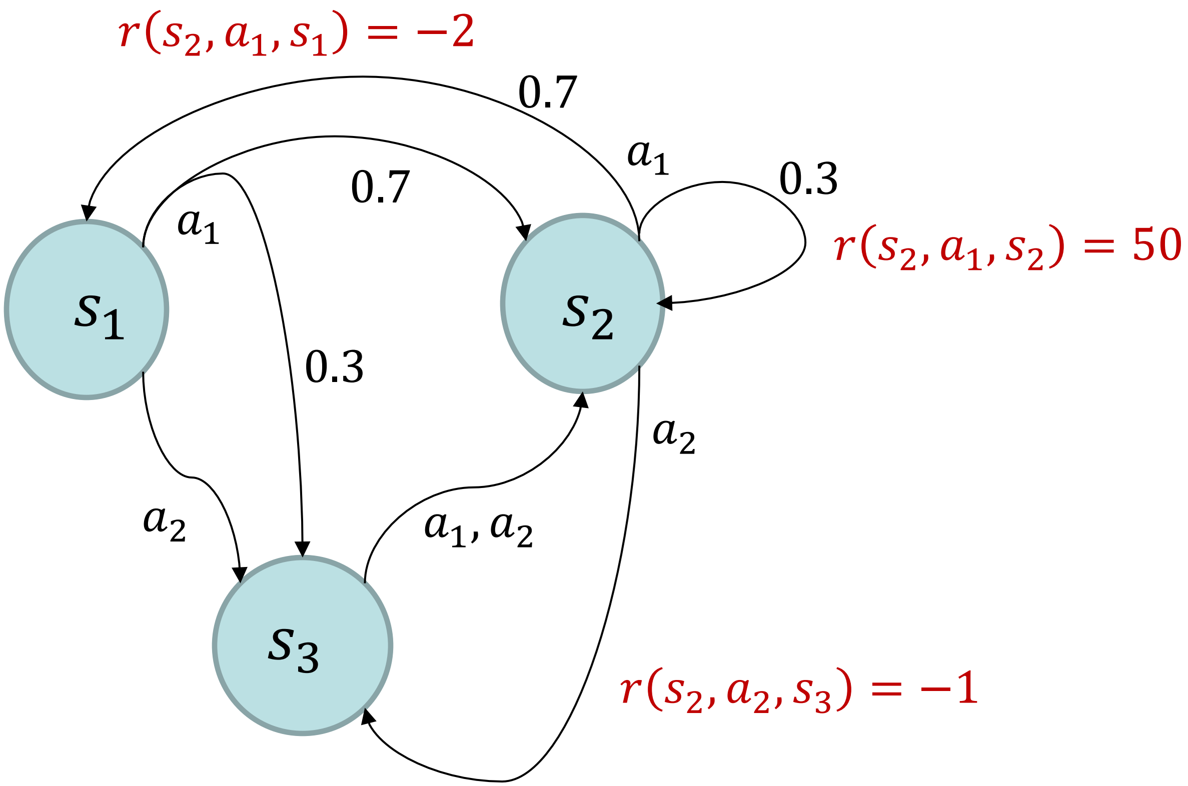
$$V^\pi(s) = \sum_a \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^\pi(s')]$$

$$V_{k+1}^\pi(s) \leftarrow \sum_a \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k^\pi(s')]$$

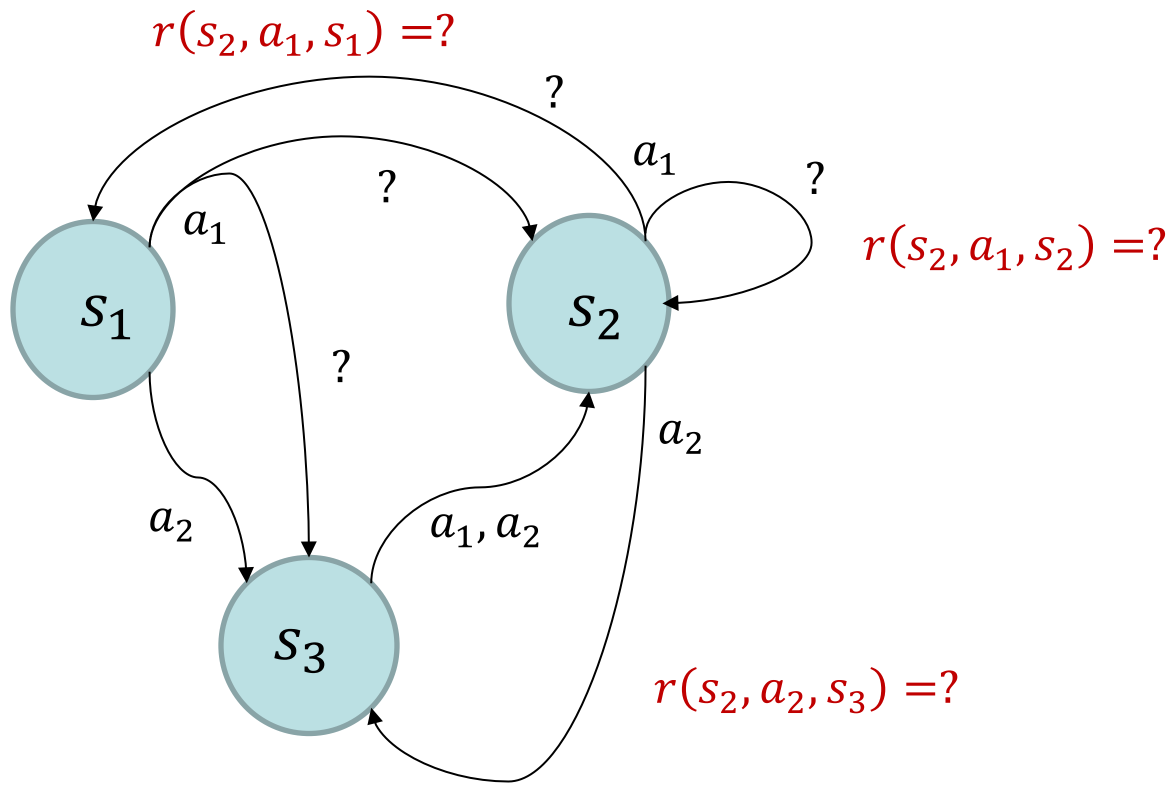
Solving MDP with VI or PI is offline planning

- The agent is given how the environment works
- The agent works out the optimal policy in its mind.
- The agent never really starts to play at all.
- No learning is happening.

State-space diagram representation of an MDP: An example with 3 states and 2 actions.



What happens if you do not know the rewards / transition probabilities?



Then you have to learn by interacting with the unknown environment.

You cannot use only offline planning!

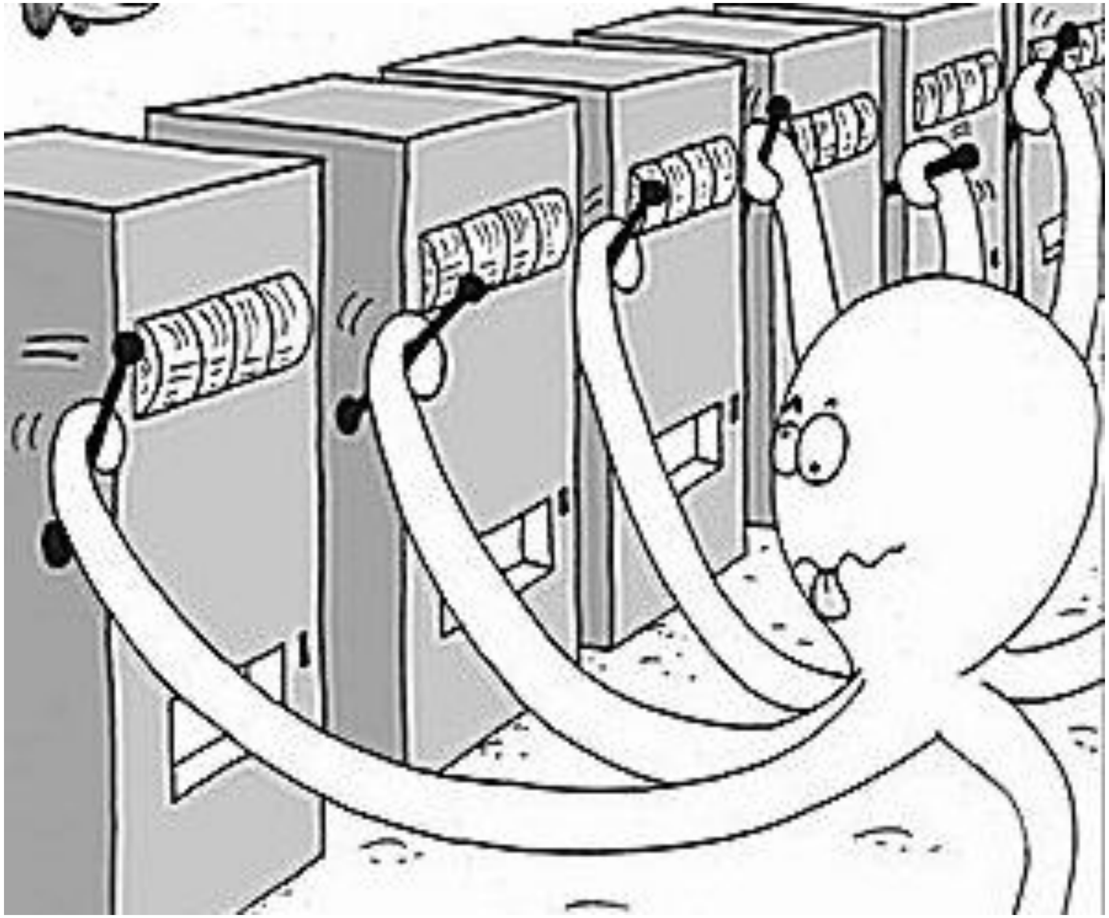
Exploration: Try unknown actions to see what happens.

Exploitation: Maximize utility using what we know.

Let us tackle different aspects of the RL problem one at a time

- Markov Decision Processes:
  - Dynamics are given no need to learn
- **Bandits: Explore-Exploit in simple settings**
  - RL without dynamics
- Full Reinforcement Learning
  - Learning MDPs

Slot machines and Multi-arm bandits



Multi-arm bandits: Problem setup

- No state. k-actions $a \in \mathcal{A} = \{1, 2, ..., k\}$
- You decide which arm to pull in every iteration

$$A_1, A_2, ..., A_T$$

- You collect a cumulative payoff of $\sum_{t=1}^{T} R_t$
- The goal of the agent is to maximize the expected payoff.
  - For future payoffs?
  - For the expected cumulative payoff?

Key differences from MDPs

- Simplified:
  - No state-transitions
- But:
  - We are not given the expected reward r(s, a, s')
  - We need to learn the optimal policy by trials-and-errors.

A 10-armed bandits example

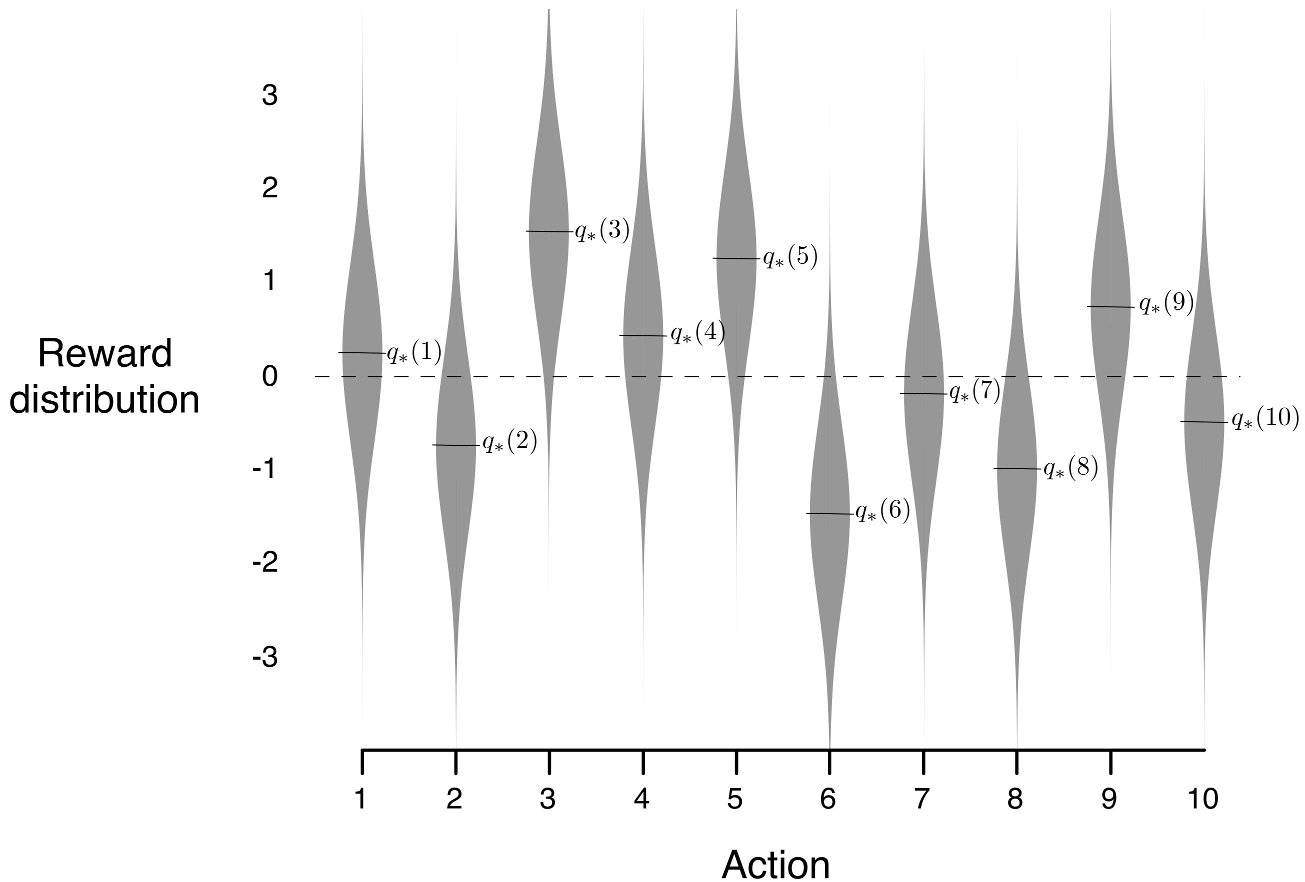


Figure 2.1: An example bandit problem from the 10-armed testbed. The true value $q_*(a)$ of each of the ten actions was selected according to a normal distribution with mean zero and unit variance, and then the actual rewards were selected according to a mean $q_*(a)$ unit variance normal distribution, as suggested by these gray distributions.

How do we measure the performance of an **online learning agent**?

- The notion of "Regret":
  - I wish I have done things differently.
  - Comparing to the best actions in the hindsight, how much worse did I do.

- For MAB, the regret is defined as follow

$$T \max_{a \in [k]} \mathbb{E}[R_t | a] - \sum_{t=1}^{T} \mathbb{E}_{a \sim \pi} \left[\mathbb{E}[R_t | a] \right]$$

Greedy strategy

- Expected reward

$$q_*(a) \doteq \mathbb{E}[R_t \mid A_t = a] \,.$$

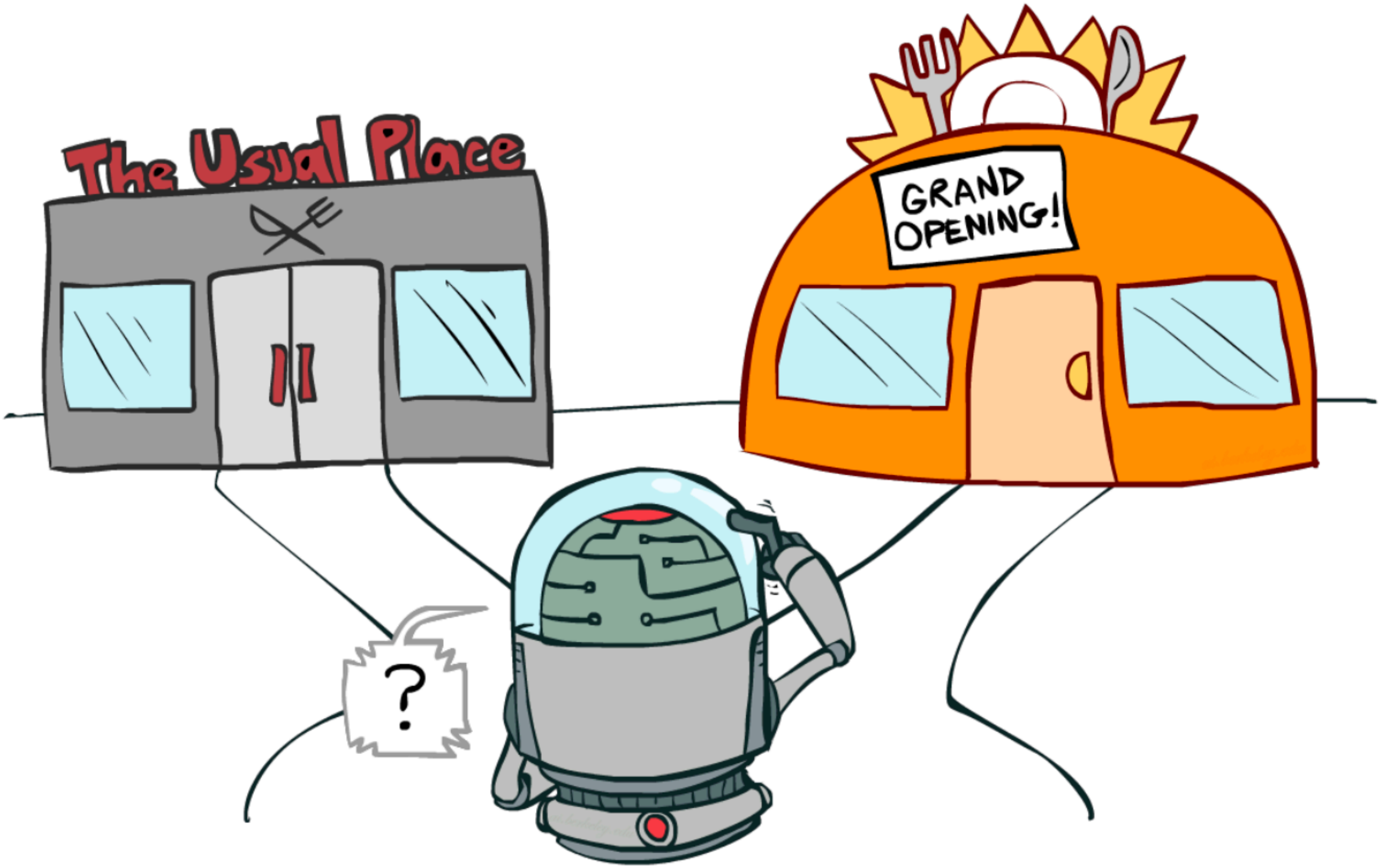
- Estimate the expected reward

$$Q_t(a) \doteq \frac{\text{sum of rewards when } a \text{ taken prior to } t}{\text{number of times } a \text{ taken prior to } t} = \frac{\sum_{i=1}^{t-1} R_i \cdot \mathbb{1}_{A_i=a}}{\sum_{i=1}^{t-1} \mathbb{1}_{A_i=a}}$$

- Choose $A_t \doteq \underset{a}{\arg\max}\, Q_t(a),$

What is the issue with this strategy?

Exploration vs. Exploitation



(Illustration from Dan Klein and Pieter Abbeel's course in UC Berkeley)

Next Tuesday

- Bandits algorithms
  - Explore-first
  - epsilon-greedy
  - Upper confidence bound