Artificial Intelligence

CS 165A

Nov 19, 2020

Instructor: Prof. Yu-Xiang Wang

Today

→ Contextual Bandits

→ Reinforcement Learning

Recap: Multi-arm bandits: Problem setup

- No state. k-actions $a \in \mathcal{A} = \{1, 2, ..., k\}$

- You decide which arm to pull in every iteration

$$A_1, A_2, ..., A_T$$

- You collect a cumulative payoff of $\sum_{t=1}^{T} R_t$

- The goal of the agent is to maximize the expected payoff.
  - For future payoffs?
  - For the expected cumulative payoff?

Recap: How do we measure the performance of an **online learning agent**?

- The notion of "Regret":
  - I wish I have done things differently.
  - Comparing to the best actions in the hindsight, how much worse did I do.

- For MAB, the regret is defined as follow

$$T \max_{a \in [k]} \mathbb{E}[R_t | a] - \sum_{t=1}^{T} \mathbb{E}_{a \sim \pi} \left[\mathbb{E}[R_t | a] \right]$$

Recap: MAB Algorithms

- Idea: Plug-in estimate of the reward value

$O(\log \frac{1}{\delta})$

- Greedy: Regret = O(T)

w.h.p. $1-\delta$

- Explore-first: Regret = O($T^{2/3}$)

$r(a) \in [\hat{R}(a) - O(\frac{1}{\sqrt{n_a}}), \hat{R}(a) + O(\frac{1}{\sqrt{n_a}})]$

- epsilon-greedy: Regret = O($T^{2/3}$)

$O(\sqrt{\frac{1}{n}})$

- Upper Confidence Bound: Regret = O($T^{1/2}$)
  - Optimal in the sense that no algorithm can do better

Recap: Upper Confidence Bound algorithm (UCB)

- At time t, choose the action

$0 \le \gamma \le 1$

$+\infty$

$$A_t \leftarrow \underset{a}{\operatorname{argmax}} \left[Q_t(a) + c\sqrt{\frac{\log(1+t)}{N_t(a)}} \right]$$

- Idea: Be optimistic
  - Choose an option that maximizes the upper confidence bound.

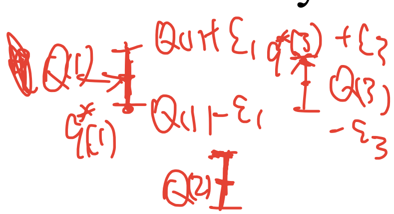
$$\mathbb{E}[\text{Regret}] = O(\sqrt{Tk})$$

- The proof is out of the scope of this course. For those who are interested, please look up. It's not difficult.

Idea of the analysis of UCB

- Design principle: Optimistic in the face of uncertainty

- Idea why UCB improves over random exploration:
  - When you follow the UCB approach, the maximum regret that you can incur in each iteration is the confidence interval of the arm you pick. (why is that?)
  - Exploration will be restricted to those arms that are not "eliminated" yet.



$$\text{Regret} = q^*(3) - q^*(1) \le \hat{Q}(3) + \varepsilon_3 - (\hat{Q}(1) - \varepsilon_1) \le \hat{Q}(1) + \varepsilon_1 - (\hat{Q}(1) - \varepsilon_1) = 2\varepsilon_1$$

- In other words, UCB explore and exploit at the same time!

Intuitively why is the O($T^{1/2}$) regret optimal?

- Consider a 2-arm bandit problem and two parallel worlds:
  - Arm 1 has expected reward 0.5, Arm 2 has 0.5 + eps
  - Arm 1 has expected reward 0.5+eps, Arm 2 has 0.5
  - Reward distribution is Bernoulli distribution.

- Set eps = O($1/\sqrt{T}$). Recall that you need to pull Arm 1 and Arm 2 both for $\Omega(T)$ times in order to identify which one is better. Thus the regret needs to be $\Omega(T \times \frac{1}{\sqrt{T}})$.
- To say it differently: If any algorithm is able to achieve better regret, then it implies an estimator that estimates the p of a biased coin with fewer samples than required. Thus a contradiction.

A 10-armed bandits benchmark

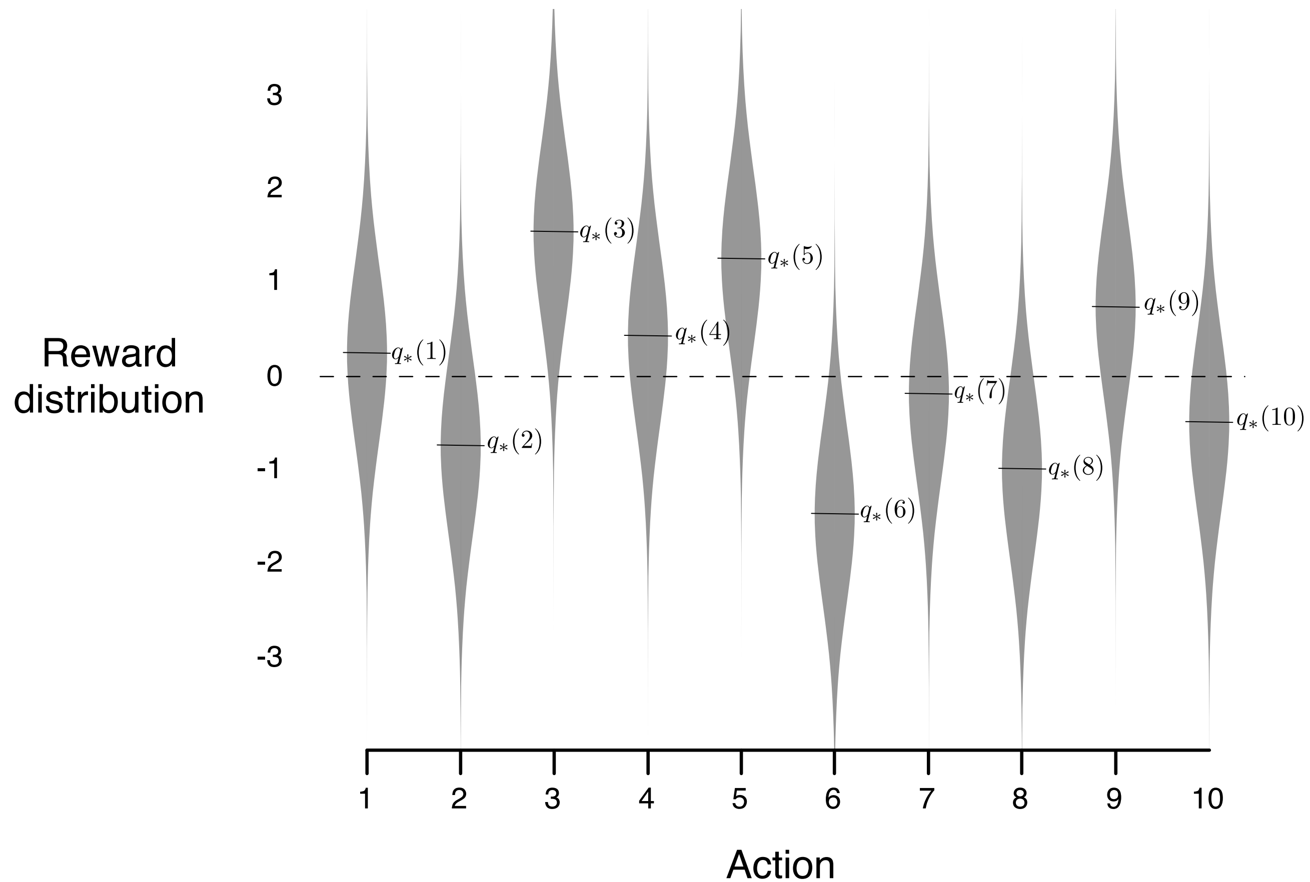
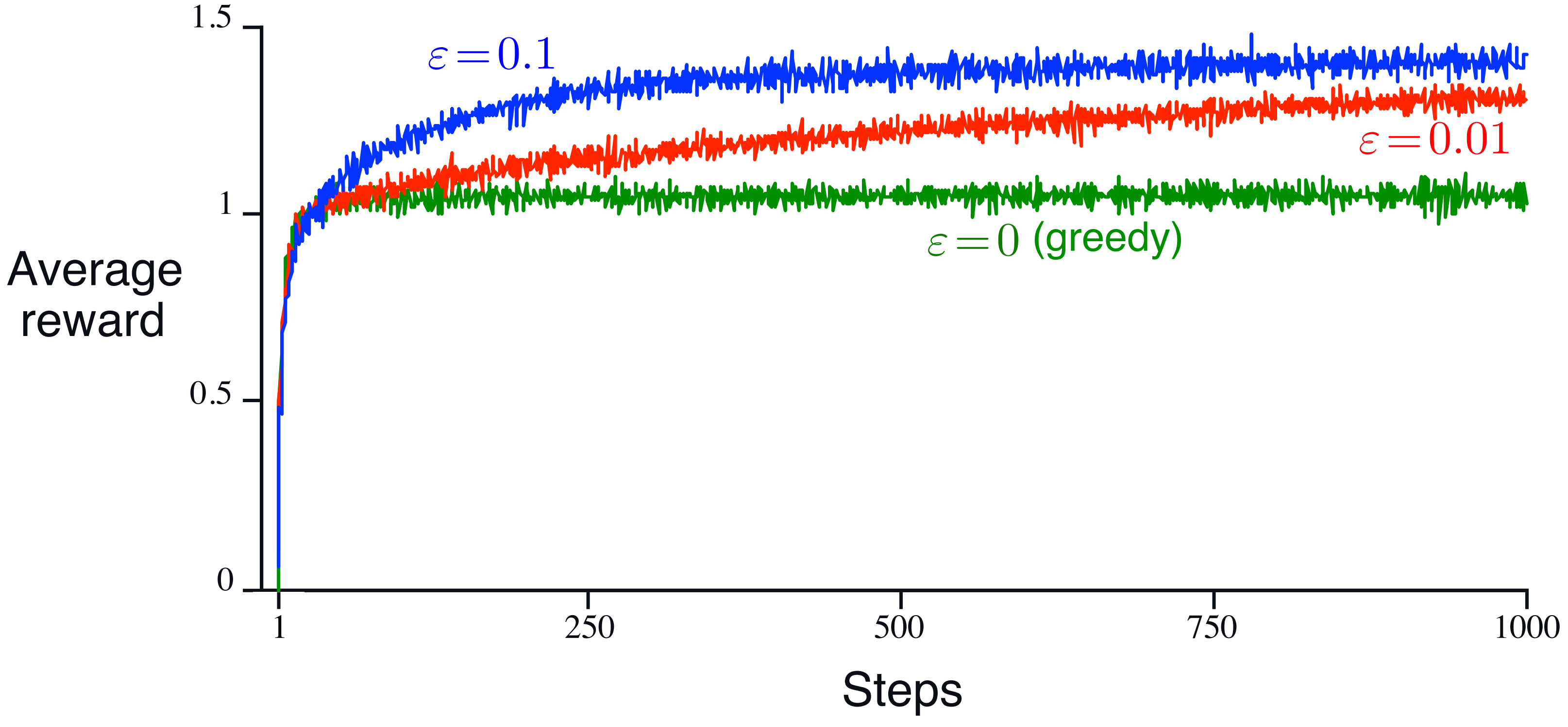
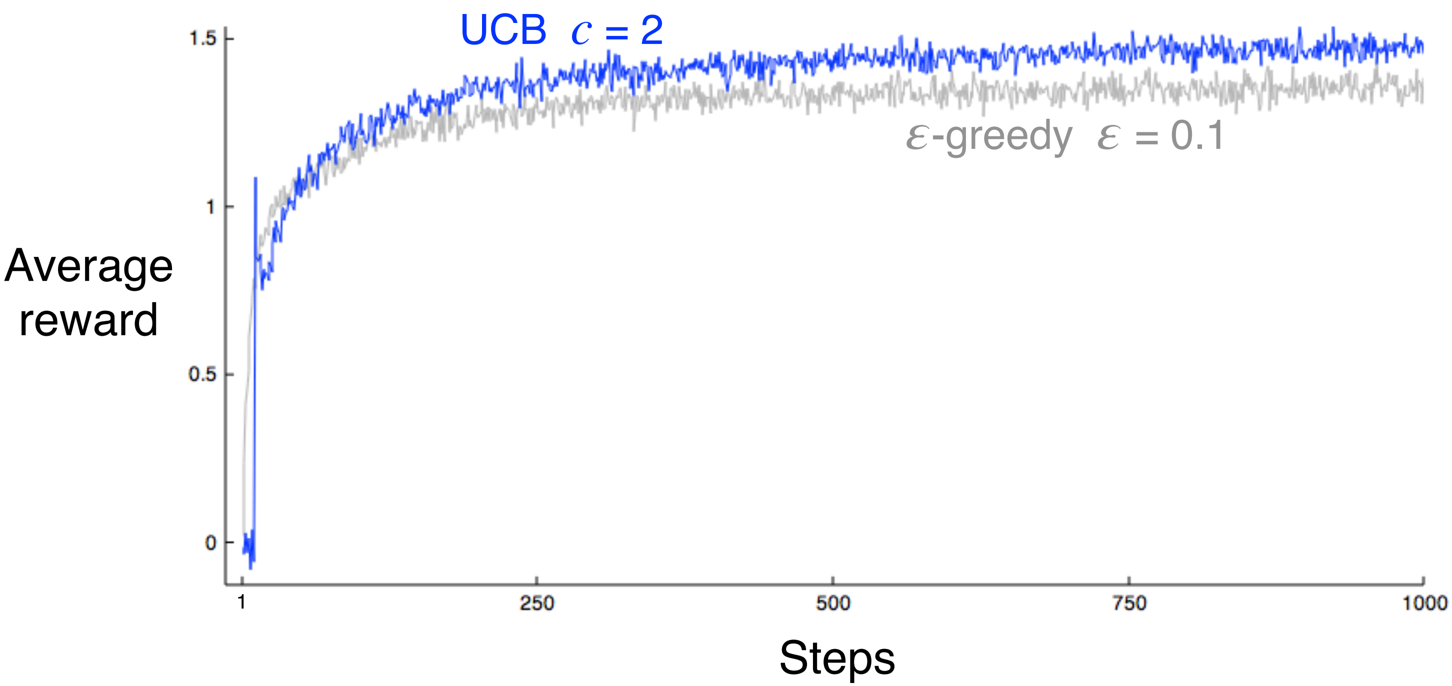


Figure 2.1: An example bandit problem from the 10-armed testbed. The true value $q_*(a)$ of each of the ten actions was selected according to a normal distribution with mean zero and unit variance, and then the actual rewards were selected according to a mean $q_*(a)$ unit variance normal distribution, as suggested by these gray distributions.

Comparing the different algorithms



UCB vs. ε-Greedy



Variants of Bandits problems

- Online Learning from Expert Advice
  - Adversarial chooses the outcome
  - You observe outcome of other arms as well
  - Compare against the best arm in the hindsight

- Adversarial k-Armed Bandits
  - Same as above. But you observe only your arm.

- Nonstationary Bandits
  - Stochastic but the reward distribution changes over time.
  - Compare against the best arm for each time.

- Contextual bandits: you have a state in each time point.

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Remark: In all these problems, there are algorithms with provably low-regret.

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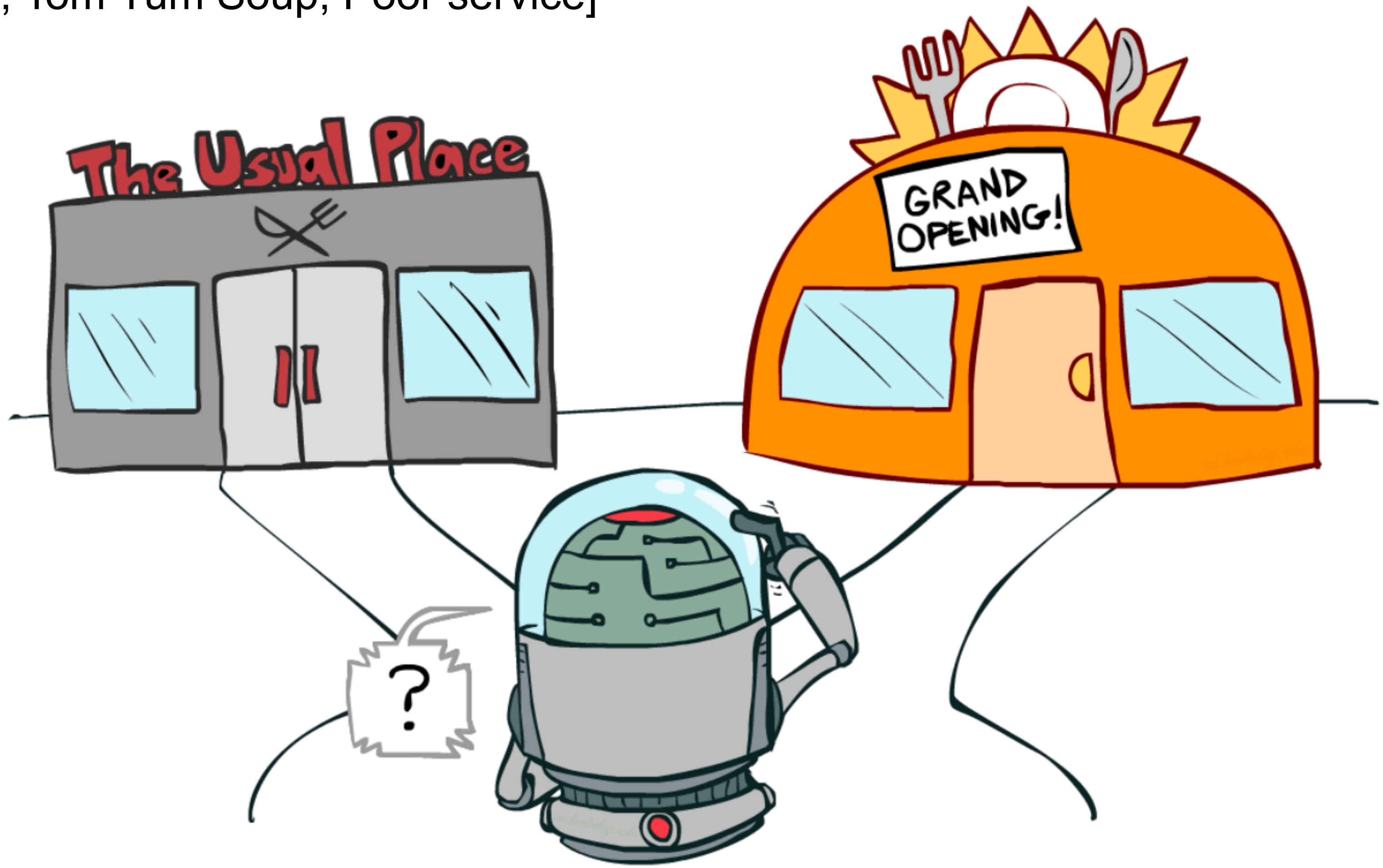
Do I have to try, if I have features?

Features:

[Noodles, Tom Yum Soup, Poor service]

Features:

[Burger, Fries, Onion Ring, Fried Chicken]



(Illustration from Dan Klein and Pieter Abbeel's course in UC Berkeley)

We know how to use with features, don’t we?

- Classifier agent
  - Take features of a restaurant as input
  - Output a prediction of “will I like the food?”
- Train with supervised learning
  - Using the my previous visits to the restaurants
  - Using Yelp reviews

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How to explore?

Contextual Bandits: Problem Setup

- For each round $t = 1, 2, 3, \ldots, T$:
  - A context $x_t \sim$ unknown distribution i.i.d.
  - Agent picks an action $a_t = 1,2,3,..,K$
  - Reward $r_t \sim D(. \mid x_t, a_t)$

- Agent's goals:
  - Learn the best policy out of many policies Π
  - Minimize the cumulative regret

A finite family of policies

$$T \cdot \max_{\pi \in \Pi} \mathbb{E}_{\pi}[r_t(x_t, a_t)] - \mathbb{E}_{\text{Agent's policy}}\left[\sum_{t=1}^{T} r_t(x_t, a_t)\right]$$

Reward from the best policy

Reward collected by the Agent

Applications of Contextual Bandits

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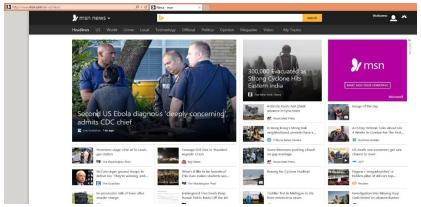
Personalized news?



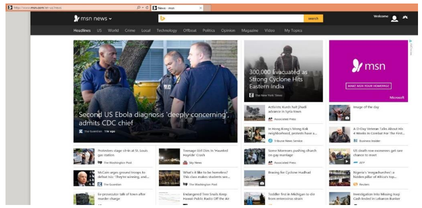
Repeatedly:

1. Observe features of user+articles
2. Choose a news article.
3. Observe click-or-not

Goal: Maximize fraction of clicks

Applications of Contextual Bandits

Personalized news?



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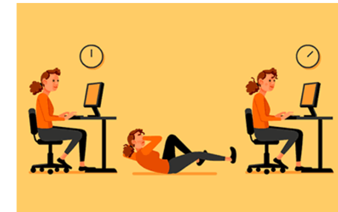
Health advice?



Repeatedly:

1. Observe features of user+advice
2. Choose an advice.
3. Observe steps walked

Goal: Healthy behaviors (e.g. step count)

Applications of Contextual Bandits

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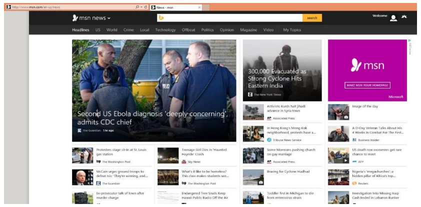

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Health advice?

Repeatedly:

1. Observe features of user+advice
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Goal: Healthy behaviors (e.g. step count)

amazon

Recommendations

buy or not buy

Exploration vs. Exploitation in Contextual Bandits.

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  - Infinite state space, never see the same context again.
  - Exponentially large policy space

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- Challenging because:
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- Ideas:
  - ExploreFirst, ε-Greedy $O(T^{2/3})$
  - UCB? But how do we construct Confidence Interval for an exponentially large set of policies?
- Optimal regret:

$$O(\sqrt{KT\log|\Pi|})$$

O(d)

Remainder of the lecture today

- Reinforcement learning for MDPs
  - Model-based vs model-free algorithms
  - Online policy iterations
  - Temporal difference learning

- Readings:
  - AIMA Ch. 21.1-21.3 (Ch 22.1- 22.3 in 4th Edition)
  - Sutton and Barto: Ch 4-6
  - Maybe: Sutton and Barto: Ch 6, Ch 13

Let us tackle different aspects of the RL problem one at a time

- Markov Decision Processes:
  - Dynamics are given no need to learn
- Bandits: Explore-Exploit in simple settings
  - RL without dynamics
- **Full Reinforcement Learning**
  - Learning MDPs

Recap: Tabular MDP

- Discrete State, Discrete Action, Reward and Observation

$$S_t \in \mathcal{S} \quad A_t \in \mathcal{A} \quad R_t \in \mathbb{R} \quad \sout{O_t \in \mathcal{O}}$$

- Policy:
  - When the state is observable: $\pi : \mathcal{S} \to \mathcal{A}$
  - ~~Or when the state is not observable~~

$$\sout{\pi_t : (\mathcal{O} \times \mathcal{A} \times \mathbb{R})^{t-1} \to \mathcal{A}}$$

- Learn the best policy that maximizes the expected reward
  - Finite horizon (episodic) RL: $\pi^* = \arg\max_{\pi \in \Pi} \mathbb{E}[\sum_{t=1}^{T} R_t]$ T: horizon
  - Infinite horizon RL: $\pi^* = \arg\max_{\pi \in \Pi} \mathbb{E}[\sum_{t=1}^{\infty} \gamma^{t-1} R_t]$

γ: discount factor

Recap: Policy Iterations and Value Iterations

- What are these algorithms for?
  - Algorithms of computing the V* and Q* functions from MDP parameters

- Policy Iterations

$$\pi_0 \rightarrow^E V^{\pi_0} \rightarrow^I \pi_1 \rightarrow^E V^{\pi_1} \rightarrow^I \ldots \rightarrow^I \pi^* \rightarrow^E V^*$$

- Value iterations

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k(s')]$$

- How do we make sense of them?
  - Recursively applying the Bellman equations until convergence.

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*These methods are called "Dynamic Programming" approaches in Chap 4 of Sutton and Barto.

Revisit the dynamic programming approach

- Policy Evaluation

$$V_{k+1}^{\pi}(s) \leftarrow \sum_{a} \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k^{\pi}(s')]$$

- Policy improvement

$$\begin{aligned} \pi'(s) &= \arg\max_a Q^{\pi}(s,a) \\ &= \arg\max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k^{\pi}(s')] \end{aligned}$$

- Value iterations

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k(s')]$$

Revisit the dynamic programming approach

- Policy Evaluation

$$V^{\pi}_{k+1}(s) \leftarrow \sum_a \pi(a|s) \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^{\pi}_k(s')]$$

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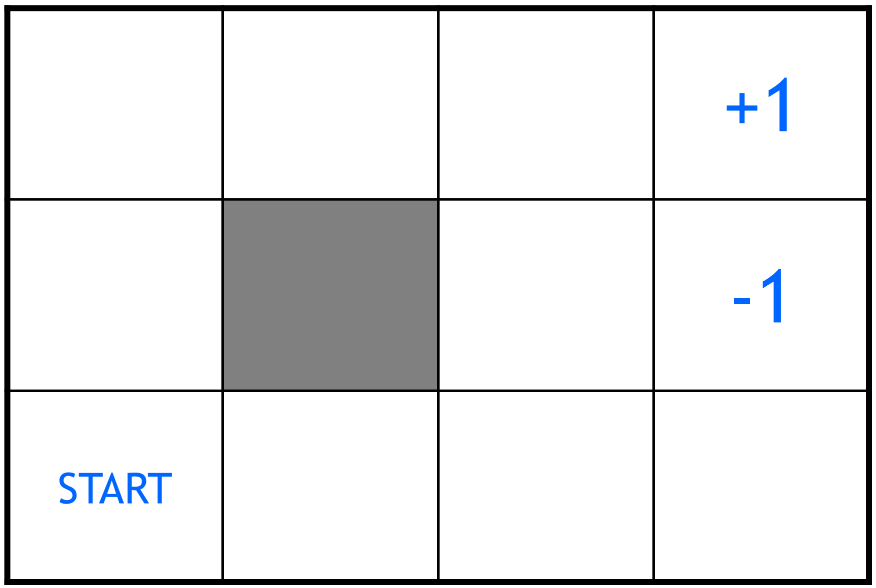
- Value iterations

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k(s')]$$

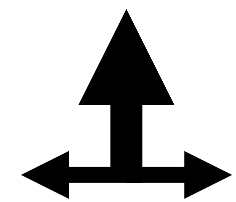
***We do not have the MDP parameters in RL!**

Example: Robot in a room.



actions: UP, DOWN, LEFT, RIGHT

UP



80%	move UP
10%	move LEFT
10%	move RIGHT

- reward +1 at [4,3], -1 at [4,2]
- reward -0.04 for each step
- what's the strategy to achieve max reward?

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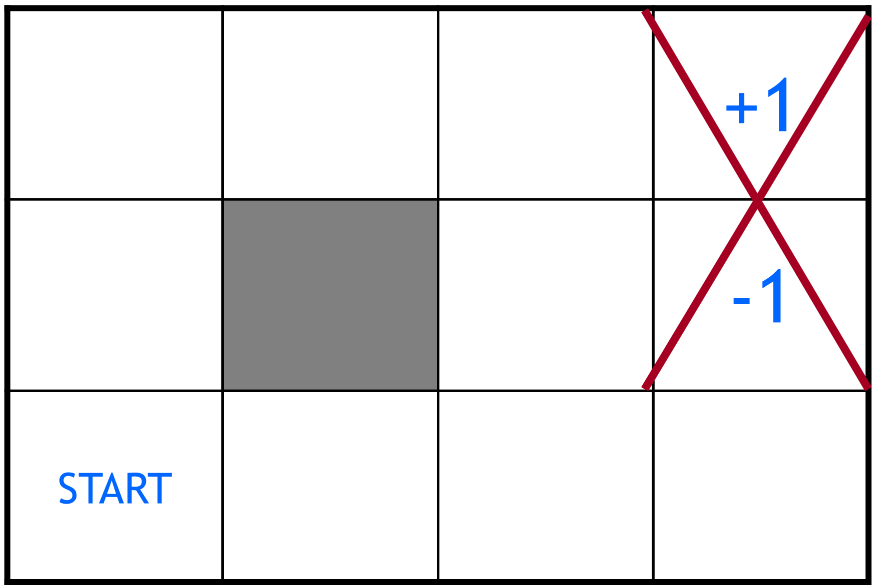

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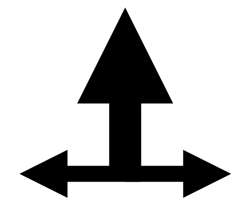

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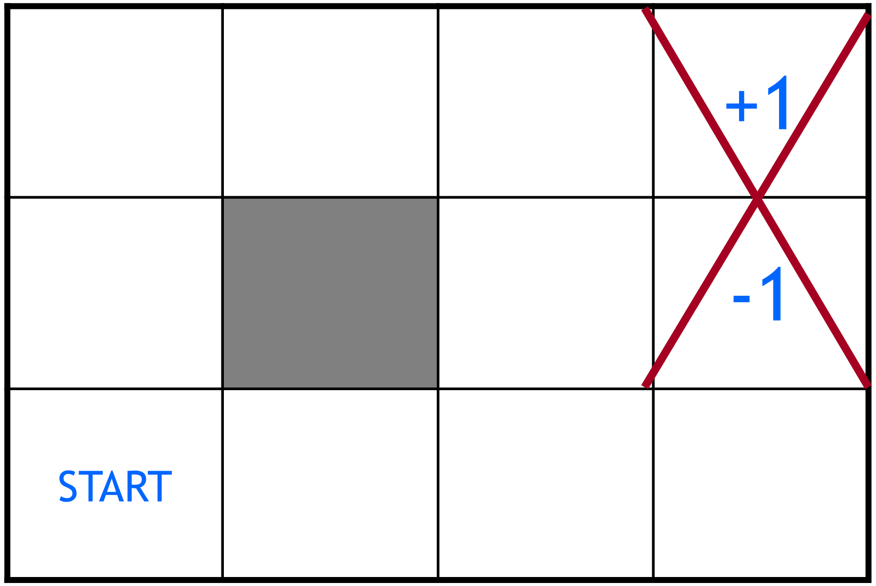

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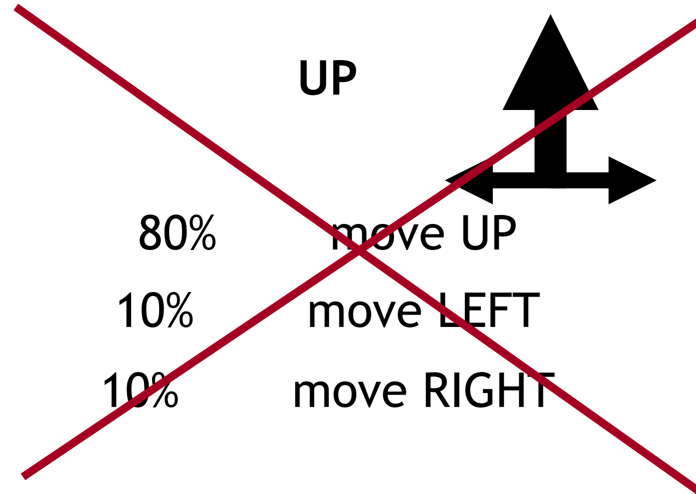

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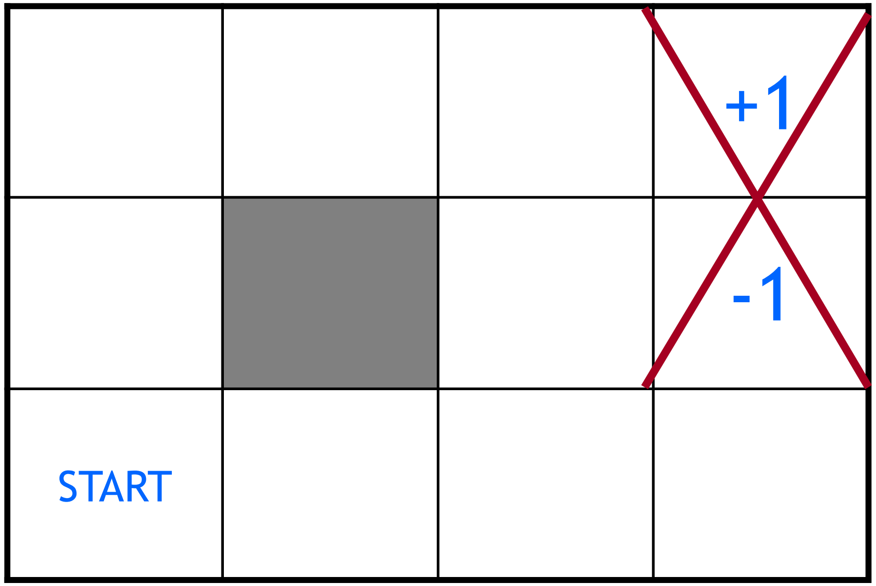
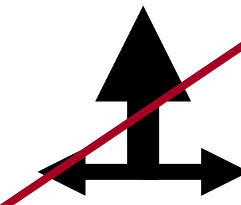


Action 1, Action 2, Action 3, Action 4

actions: ~~UP, DOWN, LEFT, RIGHT~~

~~**UP**~~



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~~10% move RIGHT~~

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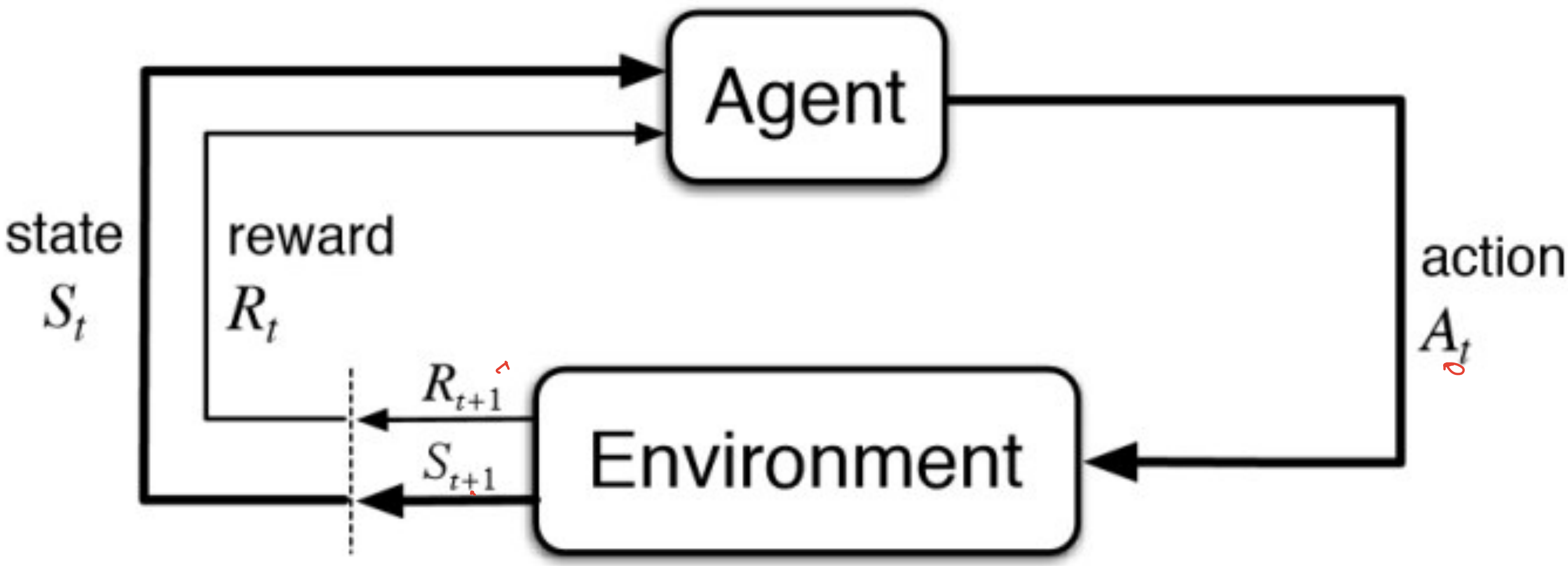
Instead, reinforcement learning agents have "online" access to an environment

- State, Action, Reward
- Unknown reward function, unknown state-transitions.
- Agents can "act" and "experiment", rather than only doing offline planning.



Idea 1: **Model-based** Reinforcement Learning

- Model-based idea
  - Let's approximate the model based on experiences
  - Then solve for the values as if the learned model were correct

- **Step 1**: Get data by running the agent to explore
  - Many data points of the form: $\{(s_1, a_1, s_2, r_1), \ldots, (s_N, a_N, s_{N+1}, r_N)\}$

- **Step 2**: Estimate the model parameters
  - $\hat{P}(s'|s,a)$ --- again this is a CPT we need to observe the transition many times for each s, a
  - $\hat{r}(s', a, s)$ --- this is an estimate of the empirical rewards.

Then we can plug in these estimates and then use dynamic programming for policy evaluation / improvements.

$$V_{k+1}^{\pi}(s) \leftarrow \sum_a \pi(a|s) \sum_{s'} \hat{P}(s'|s,a)[\hat{r}(s,a,s') + \gamma V_k^{\pi}(s')]$$

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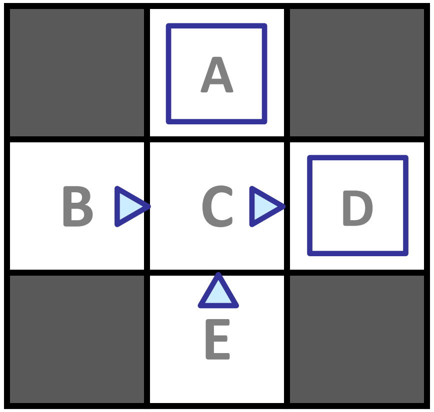
* These iterations will produce $\hat{V}^*$ and $\hat{Q}^*$ functions, and then $\hat{\pi}^*$

Example: Model-Based RL (2 min exercise)

Input Policy π



Assume: γ = 1

Observed Episodes (Training)

Learned Model

$\hat{P}(s'|s,a)$

$\hat{r}(s,a,s')$

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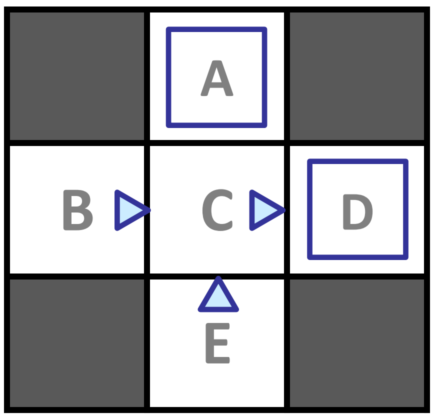

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Observed Episodes (Training)

Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 3

E, north, C, -1
C, east, D, -1
D, exit, x, +10

Episode 4

E, north, C, -1
C, east, A, -1
A, exit, x, -10

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$\hat{P}(s'|s,a)$

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Slide credits: Pat and Stephanie

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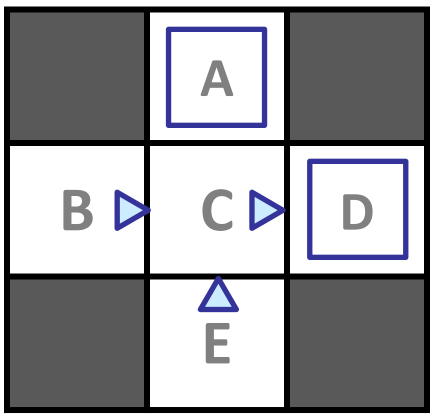

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Episode 3

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C, east, D, -1
D, exit, x, +10

Episode 4

E, north, C, -1
C, east, A, -1
A, exit, x, -10

Learned Model

$$\hat{P}(s'|s,a)$$

T(B, east, C) = $\frac{2}{2} = 1$
T(C, east, D) = $\frac{3}{4}$
T(C, east, A) = $\frac{1}{4}$
…

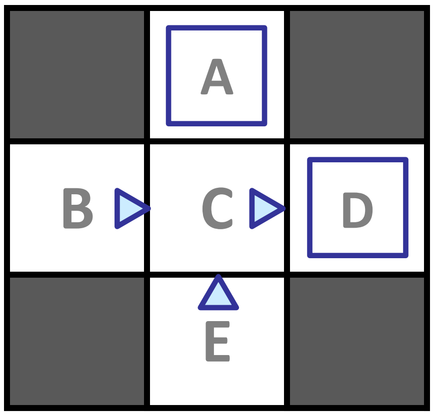
$$\hat{r}(s,a,s')$$

R(B, east, C) = $\frac{-1-1}{2} = -1$
R(C, east, D) = -1
R(D, exit, x) = $\frac{10+10+10}{3} = 10$
…

Slide credits: Pat and Stephanie

Example: Model-Based RL (2 min exercise)

Input Policy π



Assume: γ = 1

Observed Episodes (Training)

Learned Model

$$\hat{P}(s'|s,a)$$

$$\hat{r}(s,a,s')$$

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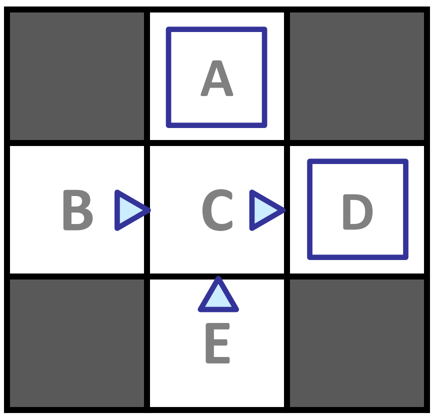

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Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 3

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C, east, D, -1
D, exit, x, +10

Episode 4

E, north, C, -1
C, east, A, -1
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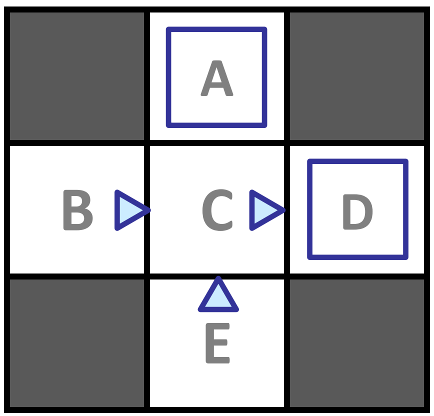

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A, exit, x, -10

Learned Model

$$\hat{P}(s'|s,a)$$

T(B, east, C) = 1.00
T(C, east, D) = 0.75
T(C, east, A) = 0.25
…

$$\hat{r}(s,a,s')$$

R(B, east, C) = -1
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  - **Question: What is an example of this?**

*Need to somehow update the "exploration policy" on the fly!

More caveats

- The fitted model is just an approximation of the environment.
- How does the error in the fitted MDP translate into the error in the estimated value functions V* and Q*?

$\|Q^* - \hat{Q}^*\|_\infty \leq \varepsilon$

- How does the error in the estimated Q* function affect the suboptimality of the policy that maximizes $\hat{Q}^*$?
- Answered by "Simulation Lemma" (Kearns and Singh, 2002)
  - Resurgence of research on this more recently: Yin and W. (2020), Yin, Bai and W. (2020)

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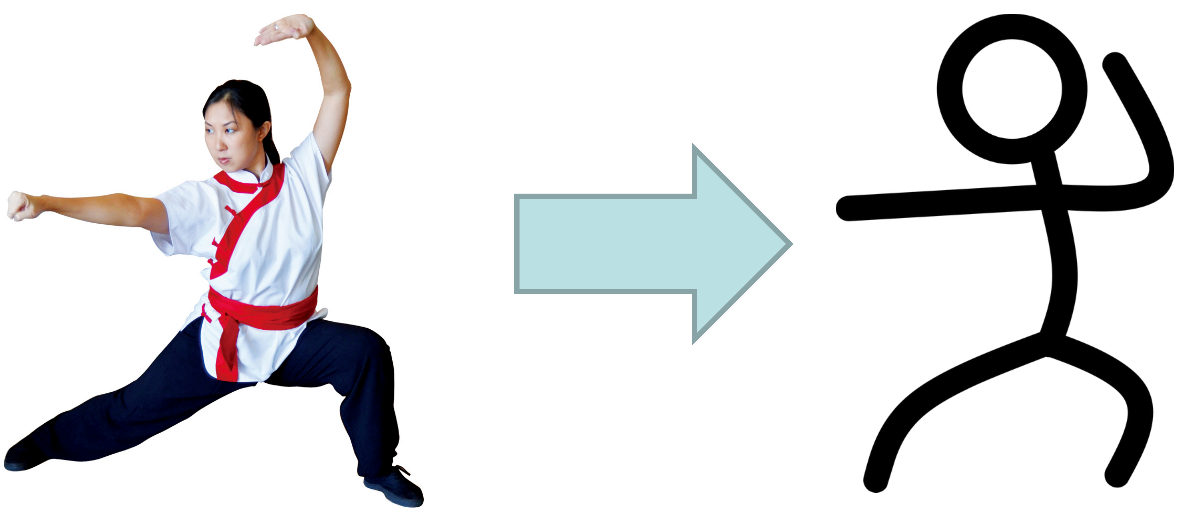

Idea 2: **Model-free** Reinforcement Learning

- Do we need the model? Can we learn the Q function directly?

$$\pi_0 \rightarrow^E V^{\pi_0} \rightarrow^I \pi_1 \rightarrow^E V^{\pi_1} \rightarrow^I \ldots \rightarrow^I \pi^* \rightarrow^E V^*$$

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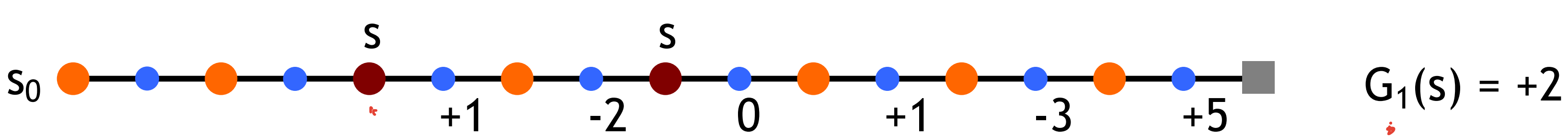
  - Maybe we can do policy evaluation without estimating the model?

Monte Carlo Policy Evaluation (Prediction)

- want to estimate $V^{\pi}(s)$
  = expected return starting from s and following π
  – estimate as average of observed returns in state s
- We can execute the policy π
- first-visit MC
  – average returns following the first visit to state s



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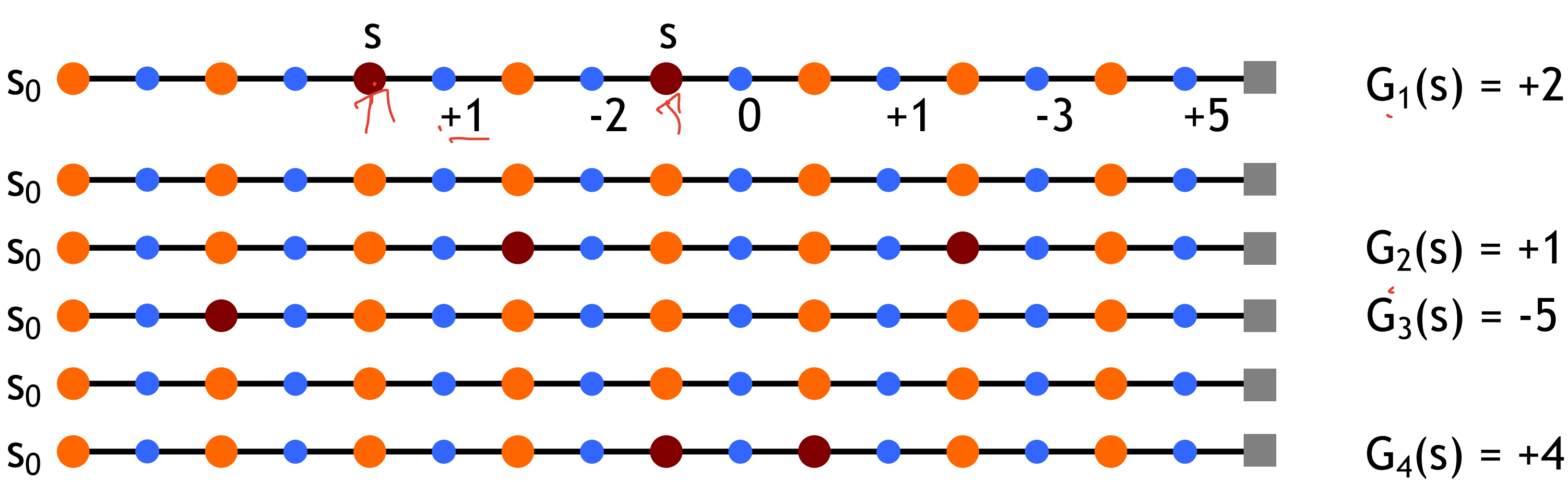
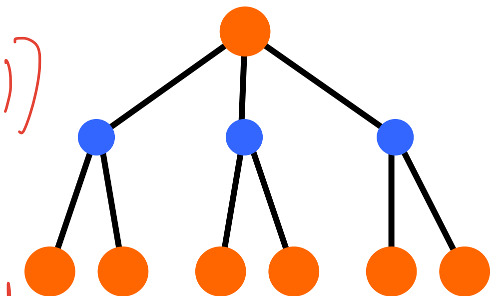


$V^{\pi}(s) \approx (2 + 1 - 5 + 4)/4 = 0.5$

Monte Carlo Policy Optimization (Control)



- V^{π} not enough for policy improvement
  - need exact model of environment
- estimate $Q^{\pi}(s,a)$

$$\pi'(s) = \arg\max_a Q^{\pi}(s,a)$$

- MC control

$$\pi_0 \rightarrow^E Q^{\pi_0} \rightarrow^I \pi_1 \rightarrow^E Q^{\pi_1} \rightarrow^I \dots \rightarrow^I \pi^* \rightarrow^E Q^*$$

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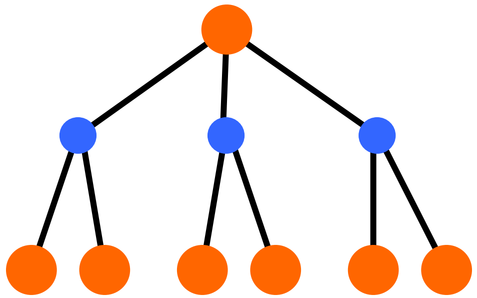

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Improved Monte-Carlo Q-function estimate using Bellman equations

- Recall:

$$Q^\pi(s,a) = \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \sum_{a'} \pi(a'|s')Q^\pi(s',a')]$$

$$Q^\pi(s,a) = r^\pi(s,a) + \gamma \mathbb{E}_{s' \sim P(s'|s,a)}[V^\pi(s')]$$

average$(R_i + \gamma V^\pi(s_i'))$

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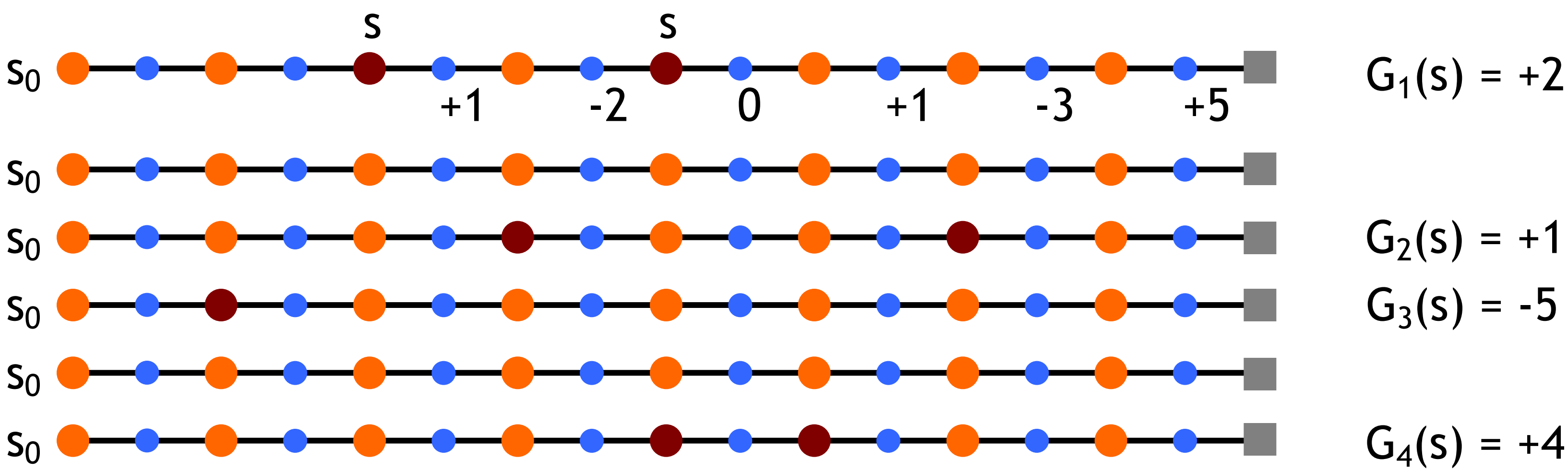
*No need to estimate P(s' | s,a) or r(s,a,s') as intermediate steps.

*Require only O(SA) space, rather than O(S^2A)

Online averaging representation of MC



$V^\pi(s) \approx (2 + 1 - 5 + 4)/4 = 0.5$

- Alternative, *online averaging* update

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[G_t - V(S_t)\Big], \quad \text{where } \alpha = 1/N_{S_t}$$

$(1-\alpha)V(S_t) + \alpha G_t$

DP + MC = Temporal Difference Learning

- Monte Carlo

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Bootstrapping!

- TD-Policy evaluation

$$V(S_t) \leftarrow V(S_t) + \alpha \Big[R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \Big]$$

Bootstrap's origin

- "The Surprising Adventures of Baron Münchausen"
  - Rudolf Erich Raspe, 1785

- In statistics: Brad Efron's resampling methods
- In computing: Booting…
- In RL: It simply means TD learning

TD policy optimization (TD-control)

- SARSA (On-Policy TD-control)
  - Update the Q function by bootstrapping Bellman Equation

$$Q(S, A) \leftarrow Q(S, A) + \alpha\left[R + \gamma Q(S', A') - Q(S, A)\right]$$

  - Choose the next A' using Q, e.g., eps-greedy.

- Q-Learning (Off-policy TD-control)
  - Update the Q function by bootstrapping Bellman Optimality Eq.

$$Q(S, A) \leftarrow Q(S, A) + \alpha\left[R + \gamma \max_a Q(S', a) - Q(S, A)\right]$$

  - Choose the next A' using Q, e.g., eps-greedy, or any other policy.

Remarks:

- These are **proven to converge** asymptotically.
- Much more data-efficient in practice, than MC.
- Regret analysis is still active area of research.

Advantage of TD over Monte Carlo

- Given a trajectory, a roll-out, of T steps.
  - MC updates the Q function only once
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Remark: This is the same kind of improvement from Gradient Descent to Stochastic Gradient Descent (SGD).

The problem of large state-space is still there

- We need to represent and learn SA parameters in Q-learning and SARSA.

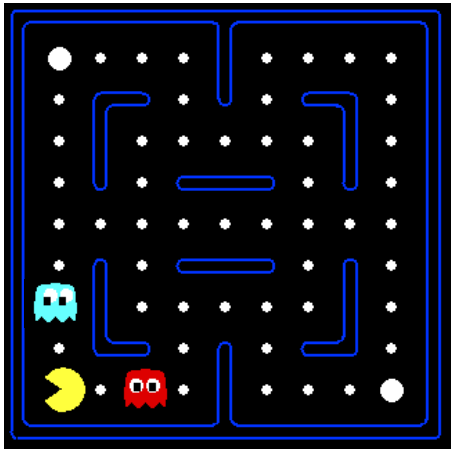
- S is often large
  - 9-puzzle, Tic-Tac-Toe: 9! = 362,800, S^2 = 1.3*10^11
  - PACMAN with 20 by 20 grid. S = O(2^400), S^2 = O(2^800)

- O(S) is not acceptable in some cases.

- Need to think of ways to "generalize"/share information across states.

Example: Pacman

Let's say we discover through experience that this state is bad:



(From Dan Klein and Pieter Abbeel)

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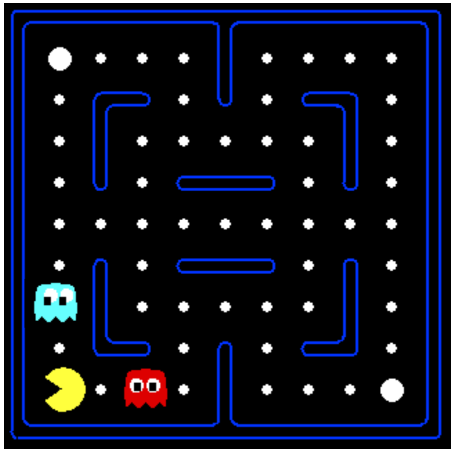
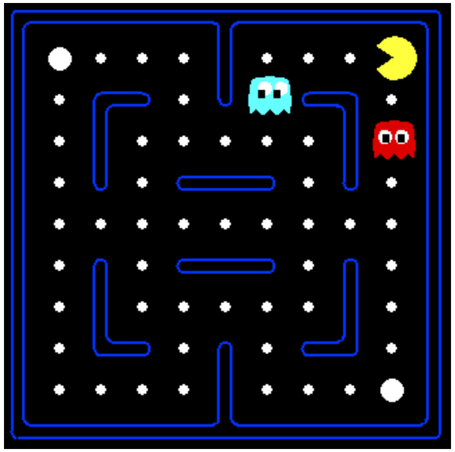


In naïve q-learning, we know nothing about this state:



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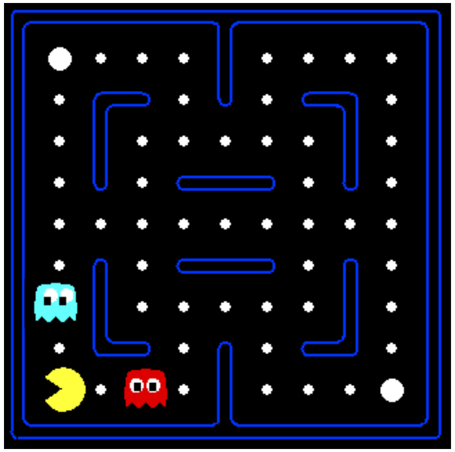

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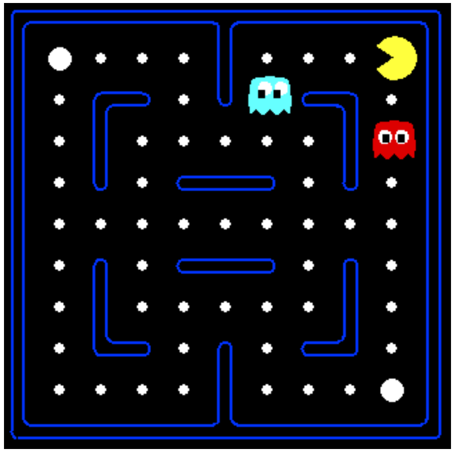
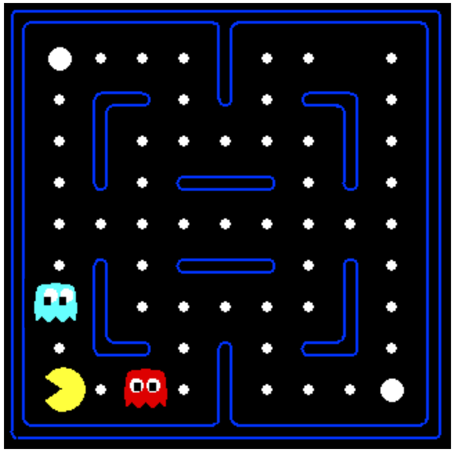


Or even this one!



(From Dan Klein and Pieter Abbeel)

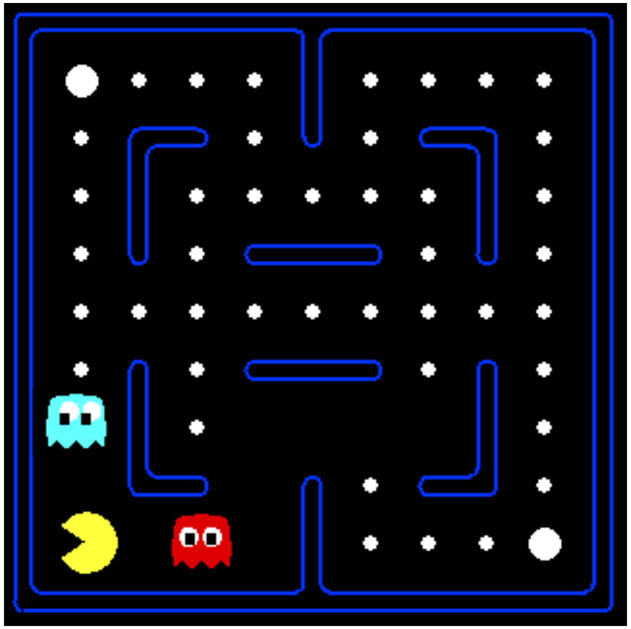
Video of Demo Q-Learning Pacman – Tiny – Watch All

Video of Demo Q-Learning Pacman – Tiny – Silent Train

Video of Demo Q-Learning Pacman – Tricky – Watch All

Why not use an evaluation function?
A Feature-Based Representations

- Solution: describe a state using a vector of features (properties)
  - Features are functions from states to real numbers (often 0/1) that capture important properties of the state
  - Example features:
    - Distance to closest ghost
    - Distance to closest dot
    - Number of ghosts
    - 1 / (dist to dot)^2
    - Is Pacman in a tunnel? (0/1)
    - …… etc.
    - Is it the exact state on this slide?
  - Can also describe a q-state (s, a) with features (e.g. action moves closer to food)



Linear Value Functions

- Using a feature representation, we can write a q function (or value function) for any state using a few weights:
  - $V_{\mathbf{w}}(s) = w_1 f_1(s) + w_2 f_2(s) + \ldots + w_n f_n(s)$
  - $Q_{\mathbf{w}}(s,a) = w_1 f_1(s,a) + w_2 f_2(s,a) + \ldots + w_n f_n(s,a)$
- Advantage: our experience is summed up in a few powerful numbers
- Disadvantage: states may share features but actually be very different in value!

Updating a linear value function

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- Original Q learning rule tries to reduce prediction error at s, a:

$$Q(s,a) \square \quad Q(s,a) + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)]$$

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- Qualitative justification:
  - Pleasant surprise: increase weights on positive features, decrease on negative ones
  - Unpleasant surprise: decrease weights on positive features, increase on negative ones

Q-Learning with function approximation

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$$\text{difference} = \left[r + \gamma \max_{a'} Q(s', a') \right] - Q(s,a)$$

Q-Learning with function approximation

$$Q(s,a) = w_1 f_1(s,a) + w_2 f_2(s,a) + \ldots + w_n f_n(s,a)$$

- Q-learning with linear Q-functions:

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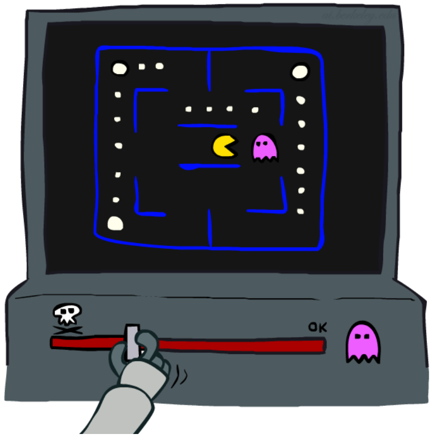

- Intuitive interpretation:
  - Adjust weights of active features
  - E.g., if something unexpectedly bad happens, blame the features that were on: disprefer all states with that state's features

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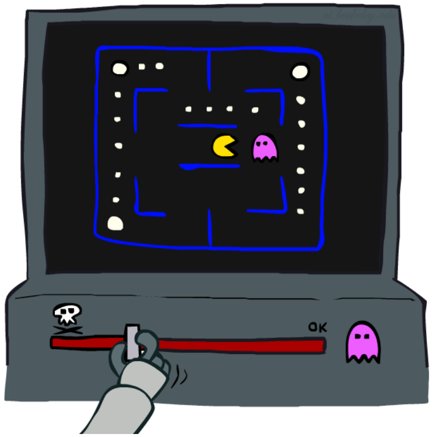

- Intuitive interpretation:
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- Formal justification: online least squares (Read the textbook!)

PACMAN Q-Learning (Linear function approx.)

So far, in RL algorithms

- Model-based approaches
  - Estimate the MDP parameters.
  - Then use policy-iterations, value iterations.
- Monte Carlo methods:
  - estimating the rewards by empirical averages
- Temporal Difference methods:
  - Combine Monte Carlo methods with Dynamic Programming
- Linear function approximation in Q-learning
  - Similar to SGD
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*Question: What is the policy class Π of interest in these methods?

Next lecture

- Wrap up RL lectures
  - Policy gradients methods
- Start logic agents / knowledge representation