INSTRUCTIONS

• *Before* you answer any questions, print your name and perm number.

• Read each question carefully. Make sure that you clearly understand each question before answering it.

• Put your answer to each question on its own page.

• You may wish to work out an answer on scratch paper before writing it on your answer page; answers that are difficult to read may lose points for that reason.

• You may not leave the room during the examination, even to go to the bathroom.

• You may not use any personal devices, such as calculators, PDAs, or cell phones.
1. (15 points) Prove or disprove the following statement: If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$, then $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

Answer

(a) Assume $M$ is a a minimal DFA for $L$ and $\overline{M}$ is not a minimal DFA for $\overline{L}$.

(b) Let $M' = (Q', \Sigma, \delta', q'_0, F')$ be a minimal DFA for $\overline{L}$.

(c) $|Q'| < |Q|$.

(d) Let $M'' = (Q', \Sigma, \delta', q'_0, Q' - F')$.

(e) $L(M'') = L$, contradicting the assumption that $M$ was a minimal DFA accepting $L$. 

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2. (10 points) The symmetric difference of 2 sets $S_1$ and $S_2$ is defined as

$$S_1 \triangle S_2 = \{ x : x \in S_1 \text{ or } x \in S_2, \text{ and } x \text{ is not in both } S_1 \text{ and } S_2 \}.$$ 

Prove that the family of regular languages is closed under symmetric difference or give a counterexample.

**Answer**

It is closed under symmetric difference.

(a) Let $S_1$ and $S_2$ be regular sets.

(b) Then

$$(S_1 \text{ or } S_2) \text{ and } (\text{ not } (S_1 \text{ and } S_2)) = (S_1 \cup S_2) \cap (\overline{S_1 \cap S_2}) = S_1 \triangle S_2$$

is regular, since regular sets are closed under union, intersection, and complement.
3. (15 points) Is there an algorithm for determining if $L_1 \subseteq L_2$, for any regular languages $L_1$ and $L_2$? Prove your answer.

**Answer**

Yes, there is. If $L_1 \subseteq L_2$ then $L_1 - L_2 = \emptyset$. An algorithm follows.

(a) Construct regular set $L_1 - L_2 = L_1 \cap \overline{L_2} = L$. This can be done since there are constructive proofs that regular sets are closed under intersection and complement.

(b) Apply the algorithm for determining if $L = \emptyset$. 

4. (15 points) Is the language \( L = \{w \in \{a, b\}^* : n_a(w) = n_b(w)\} \) regular? Prove your answer.

**Answer**

Since

- regular languages are closed under intersection
- \( L \cap a^*b^* = \{a^n b^n : n \geq 0\} \) is irregular

\( L \) is irregular.

An alternate proof that uses the Pumping Lemma follows.

(a) Assume \( L \) is regular. Then, by the Pumping Lemma, there is a natural number \( m \) such that any \( w \in L \) with \( |w| \geq m \) can be factored as \( w = xyz \) with \( |xy| \leq m \) and \( |y| > 0 \), and \( xy^iz \in L \), for \( i = 0, 1, \ldots \).

(b) Pick \( w = a^m b^m \).

(c) Then, \( a^m b^m = xyz \), where \( y = a^k \), for \( k > 0 \).

(d) By the Pumping Lemma, \( xz \in L \).

(e) But, \( n_a(xz) \neq n_b(xz) \).

(f) The assumption that \( L \) is regular thus is false.
5. (15 points) Prove that the following statement is true or prove that it is false.

If \( L_1 \) and \( L_1 \cup L_2 \) are regular languages, then \( L_2 \) is a regular language.

**Answer**

The statement is false.

Let \( L_1 = \{a, b\}^* \) and \( L_2 = \{a^n b^n : n \geq 0\} \).

Then \( L_1 \) and \( L_1 \cup L_2 \) are regular, but \( L_2 \) is irregular.
6. (10 points) Let \( L = \{ a^n b^n : n \geq 0 \} \). Is \( L^2 \) context-free? Prove your answer.

**Answer**

Yes, it is.

A CFG that recognizes \( L^2 \) is \( G_2 = (\{S_2, S\}, \{a, b\}, S_2, P) \), where \( P \) has the following productions:

\[
\begin{align*}
S_2 & \rightarrow SS, \\
S & \rightarrow aSb | \lambda.
\end{align*}
\]
7. (10 points) Is the following grammar ambiguous? Prove your answer.

\[
S \rightarrow AB \mid aaB, \\
A \rightarrow a \mid Aa, \\
B \rightarrow b.
\]

**Answer**

Yes, it is.

The word \textit{aab} has 2 different leftmost derivations:

\[
S \Rightarrow AB \Rightarrow AaB \Rightarrow aaB \Rightarrow aab \\
S \Rightarrow aaB \Rightarrow aab
\]
8. (10 points) Construct a NPDA that accepts \( \{a^n b^{2n} : n \geq 0\} \) over input alphabet \( \{a, b, c\} \).

**Answer**

\( M = (\{q_0, q_1, q_2\}, \{a, b, c\}, \{z\}, \delta, q_0, \{q_2\}) \), where \( \delta \) is given by the following diagram.

![Diagram of NPDA](image-url)