Please read the corresponding chapter before attending this lecture.

These notes are not complete. They are supplemented with figures and material that arises during the lecture in response to questions.

Please report any errors in these notes to me (cappello@cs.ucsb.edu). I’ll fix them immediately.

*Based on An Introduction to Formal Languages and Automata, 3rd Ed., Peter Linz, Jones and Bartlett Publishers, Inc.
2.3 Equivalence of DFAs & NFAs

Defn. 2.7 Finite acceptors $M_1$ and $M_2$ are equivalent if

$$L(M_1) = L(M_2).$$

For example, the DFA of Fig. 1 is equivalent to the NFA in Fig. 2.

Figure 1: A DFA that accepts $\{01^n : n \geq 0\}$.

Figure 2: An NFA that accepts $\{01^n : n \geq 0\}$.

Clearly, for every DFA there is an equivalent NFA since every DFA essentially is an NFA (be careful about the technicalities here).

It turns out that for every NFA there is an equivalent DFA. The proof of this is constructive: Given an NFA, we construct an equivalent DFA.

The principle idea is that the DFA may have exponentially more states than the NFA: If the NFA has $|Q|$ states, the constructed DFA has at
most $2^{|Q|}$ states.

Consider the NFA in Fig. 3.

Figure 3: An NFA.

**Thm. 2.2** Let NFA $M_N = (Q_N, \Sigma, \delta_N, q_0, F_N)$. There exists a DFA $M_D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ such that

$$L(M_N) = L(M_D).$$

**Proof**

We construct a transition graph for the DFA:

1. Create graph $G_D$ with start node labelled $\{q_0\}$.

2. while ( there exists a node missing an arc )

   (a) Let the node $u$ that is missing an arc be labelled $\{s_1, s_2, \ldots, s_k\}$, and the arc is for $a \in \Sigma$. 

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(b) Compute \( v = \delta_N^*(s_1, a) \cup \delta_N^*(s_2, a) \cup \ldots \cup \delta_N^*(s_k, a) \).
(c) Add node \( v \) to \( Q_D \), if it does not already exist.
(d) Add an arc labelled \( a \) from node \( u \) to node \( v \).

3. Every state in \( G_D \) whose label contains a \( q \in F_N \) is in \( F_D \).

4. If \( M_N \) accepts \( \lambda \), the initial state of \( M_D (\{q_0\}) \) is in \( F_D \).

This procedure terminates, since:

- Every iteration eliminates 1 missing arc from \( M_D \).
- \( M_D \) has at most \( 2^{Q_N} \times |\Sigma| \) arcs.

We claim that the DFA is equivalent to the NFA:

\[ w \in L(M_N) \iff w \in L(M_D). \]

It suffices to show that there is a path in \( M_N \) labelled \( w \) from \( q_0 \) to \( q_i \) if and only if there is a path in \( M_D \) labelled \( w \) from \( \{q_0\} \) to some state \( \{\ldots, q_i, \ldots\} \).
The proof is by induction on the length of the input string.

We show:

**If** there is a path in $M_N$ labelled $w$ from $q_0$ to $q_i$ **then** there is a path in $M_D$ labelled $w$ from $\{q_0\}$ to some state $\{\ldots, q_i, \ldots\}$.

The converse is left as an exercise.
Basis $|w| = 1$:
If $\delta_N^*(q_0, w)$ contains $q_i$, then, by construction of $M_D$, $\delta(\{ q_0 \}, w)$ maps to the state whose label is $\delta_N^*(q_0, w)$.

Induction hypothesis:

- Let $|w| + n$.
- If there is a path in $M_N$ labelled $w$ from $q_0$ to $q_i$ then there is a path in $M_D$ labelled $w$ from $\{ q_0 \}$ to some state $\{ \ldots, q_i, \ldots \}$. 
Induction step:

- Let $|w| + n + 1$.
- **Assume**: there is a path in $M_N$ labelled $w$ from $q_0$ to $q_i$.
- **Show**: there is a path in $M_D$ labelled $w$ from $\{q_0\}$ to some state $\{\ldots, q_i, \ldots\}$.

1. Let $w = va$, where $|v| = n$, and $a \in \Sigma$.

2. By I.S. assumption, there is a $q_j$ such that:
   - $\delta_N(q_0, v)$ contains $q_j$
   - $\delta_N(q_j, a)$ contains $q_i$.

3. Since $|v| = n$, by I.H., there is a path in $M_D$ from its start state to a state whose label contains $q_j$. 


4. Since $\delta_N(q_j, a)$ contains $q_i$, by construction of $M_D$, there is an arc labelled $a$ from all states whose label contains $q_j$ to a state whose label contains $q_i$.

5. Thus, there is a path from $M_D$’s start state to a state whose label contains $q_i$. \diamond