Chapter 3: Recursive Definitions *

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- The corresponding textbook chapter should be read before attending this lecture.
- These notes are not intended to be complete. They are supplemented with figures, and other material that arises during the lecture period in response to questions.

*Based on Theory of Computing, 2nd Ed., D. Cohen, John Wiley & Sons, Inc.
A NEW METHOD FOR DEFINING LANGUAGES

- A precise and useful way to define sets is by recursive definition.
- A recursive definition of set $S$ has 2 kinds of rules:
  - An enumeration of $S$’s base elements;
  - Rules for making new elements in $S$ from $S$’s existing elements.
Example 1

Set $S$ is defined by the following rules:

- $0, 1 \in S$
- $a, b \in S \Rightarrow a + b \in S$.

Can you characterize $S$ using English words?
Example 2

Set $S$ is defined by the following rules:

- $0, 1 \in S$
- $a, b \in S \Rightarrow a + b, a - b \in S$.

Can you characterize $S$ using English words?
Example 3

Set $S$ is defined by the following rules:

- $0, 1 \in S$
- $a, b \in S \Rightarrow a + b, a - b \in S$.
- $a, b \in S \Rightarrow \frac{a}{b} \in S$, for $b \neq 0$.

Can you characterize $S$ using English words?
Example 4

Set $S$ is defined by the following rules:

- $0, 1 \in S$
- $a, b \in S \Rightarrow a + b, a - b \in S$.
- $a, b \in S \Rightarrow a/b \in S$, for $b \neq 0$.
- $a, b \in S \Rightarrow a\sqrt{b} \in S$.

Can you prove that $\sqrt[5]{3} \sqrt[3]{5} \in S$?
Observation: \( \sqrt{3} \sqrt{5} \in S \)

Definition of \( S \)

1. \( 0, 1 \in S \)

2. \( a, b \in S \Rightarrow a + b, \sqrt[3]{b} \in S. \)

Proof of observation

1. \( 1 \in S \) (Rule 1)

2. \( 2 = 1 + 1 \in S \) (Step 1, Rule 2)

3. \( 3 = 2 + 1 \in S \) (Step 2, Rule 2)

4. \( \sqrt{3} \in S \) (Step 3, Rule 2)

5. \( 5 = 3 + 2 \in S \) (Steps 2, 3, Rule 2)

6. \( \sqrt{3} \sqrt{5} \in S \) (Steps 4, 5, Rule 2)

Can you think of an alternate proof?
Example 5

Function $f$ is defined by the following rules:

- $f(0) = 0$, $f(1) = 1$
- $f(n) = f(n - 1) + f(n - 2)$.

Have you seen this function before?
Example 6: Kleene closure revisited

Let $S$ be a language. Define $S$’s Kleene closure, denoted $S^*$, as follows:

- $\Lambda \in S^*$
- $w \in S \Rightarrow w \in S^*$
- $x, w \in S^* \Rightarrow xw \in S^*$. 
The Language of Arithmetic Expressions

Let \( \Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, -, *, /, (,) \} \).

Define arithmetic expressions, denoted \( \text{AE} \), as follows.

1. Any number (positive, negative, or zero) is in \( \text{AE} \)

2. \( x, y \in \text{AE} \Rightarrow: \)
   
   (a) \( (x) \in \text{AE} \)
   
   (b) \(-x \in \text{AE}\) (provided that \( x \) does not already start with \(-\))
   
   (c) \( x + y \in \text{AE}\) (if the first symbol of \( y \) is not + or -)
   
   (d) \( x - y \in \text{AE}\) (if the first symbol of \( y \) is not + or -)
   
   (e) \( x \times y \in \text{AE} \)
(f) \( x/y \in AE \)

(g) \( x \ast \ast y \in AE \) (our notation for exponentiation)

This definition says nothing about the value or meaning of such an expression.

What do we want \( 3 \ast \ast 3 \ast \ast 3 \) to mean?
Example 6: Java Identifiers

- Let $\text{Letter} = \{a, A, b, B, \ldots, z, Z\}$.
- Let $\text{Digit} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
- Let $\Sigma = \text{Letter} \cup \text{Digit} \cup \{\_ , \$\}$.
- Define $\text{Identifier} = \text{Letter} \ \Sigma^*$. 