CS 40: EXAMINATION 2

Department of Computer Science
University of California, Santa Barbara
Closed-Book, 50 minutes

Winter 2011

INSTRUCTIONS

• Before you answer any questions, print your name and perm number.

• Read each question carefully. Make sure that you clearly understand each question before answering it.

• Put your answer to each question on its own page.

• You may wish to work out an answer on scratch paper before writing it on your answer page; answers that are difficult to read may lose points for that reason.

• You may not leave the room during the examination, even to go to the bathroom.

• You may not use any personal devices, such as calculators, PDAs, or cell phones.
1. (9 points) Define the following terms:

the principle of mathematical induction:
That the following statement is true:

\[(P(1) \land \forall k[P(k) \rightarrow P(k + 1)]) \rightarrow \forall n P(n).\]

\([a]_R \text{ (equivalence class of } a \text{ with respect to } R):\]
\[\{b \mid a R b\}.\]

binary relation from \(A\) to \(B:\)
A subset of \(A \times B\).
2. (11 points)

Give a big-$O$ estimate, using a function of $n$ of smallest order, for the time to sequentially execute “whatThis”.

```java
void whatsThis( double[][] a, double[][] b, double[][] c, int n )
{
    // Assume that a, b, and c are n X n matrices
    int squareRootN = (int) Math.sqrt( n );
    int logN = (int) Math.log( n );

    for ( int i = 0; i < n; i++ )
    {
        for ( int j = 0; j < squareRootN; j++ )
        {
            c[i][j] = 0.0;
            for ( int k = 0; k < logN; k++ )
            {
                c[i][j] += a[i][k] * b[k][j];
            }
        }
    }
}
```

The time to execute “whatThis” is $O(n\sqrt{n}\log(n))$. 
3. (6 points) According to the Division Algorithm:

(a) \(-7 \div 10 = ?\)
(b) \(-7 \mod 10 = ?\)

The quotient is -1; the remainder is 3.
4. (9 points)

Which of these collections of subsets are partitions on the set of bit strings of length 8?

(a) the set of bit strings that contain the bit string 00, the set of bit strings that contain the bit string 01, the set of bit strings that contain the bit string 11

(b) the set of bit strings that end with 111, the set of bit strings that end with 011, the set of bit strings that end with 00

(c) the set of bit strings that have exactly $3k$ ones, where $k$ is a nonnegative integer; the set of bit strings that have exactly $3k + 1$ ones, where $k$ is a nonnegative integer; the set of bit strings that have exactly $3k + 2$ ones, where $k$ is a nonnegative integer.

Part (c) is a partition; the others are not.
5. (12 points)

Is \( (S, R) \) a poset if \( S \) is the set of all people in the world, and \( (a, b) \in R \), where \( a \) and \( b \) are people, if:

(a) \( a \) is no shorter than \( b \)?
(b) \( a = b \) or \( a \) is a descendant of \( b \)?
(c) \( a \) and \( b \) do not have a common friend?

Only Part (b) is a poset.
6. (20 points)

Find an explicit formula for $f(n)$ if $f(1) = 1$ and $f(n) = f(n - 1) + 2n - 1$, for $n \geq 2$. Prove your result using mathematical induction.

We observe the following values for $f$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1^2$</td>
</tr>
<tr>
<td>2</td>
<td>$2^2$</td>
</tr>
<tr>
<td>3</td>
<td>$3^2$</td>
</tr>
<tr>
<td>4</td>
<td>$4^2$</td>
</tr>
<tr>
<td>5</td>
<td>$5^2$</td>
</tr>
</tbody>
</table>

Conjecture: $f(n) = n^2$.

Proof by mathematical induction:

(a) Basis $n = 1$: $f(1) = 1 = 1^2$.

(b) Assume for arbitrary $n$ that $f(n) = n^2$.

(c) We show that for $n + 1$ that $f(n + 1) = (n + 1)^2$.

\[
f(n + 1) = f(n) + 2(n + 1) - 1 \quad \text{(1)}
\]
\[
= n^2 + 2(n + 1) - 1 \quad \text{(2)}
\]
\[
= n^2 + 2n + 1 \quad \text{(3)}
\]
\[
= (n + 1)^2. \quad \text{(4)}
\]

Eq. (1) is from the definition of $f(n)$; Eq. (2) follows from Eq. (1) via the induction hypothesis.
7. (25 points)

Use strong induction to show that every positive integer \( n \) can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers \( 2^0 = 1, 2^1 = 2, 2^2 = 4, \) and so on. 

[Hint: For the inductive step, separately consider the case where \( k + 1 \) is even, and where it is odd. When it is even, note that \( (k + 1)/2 \) is an integer.]

Basis \( n = 1 \): It can be written as \( 2^0 \).

Assume that for \( n \leq k \) that \( n \) can be written as the sum of distinct powers of 2.

Let \( n = k + 1 \).

Case \( n \) is odd: Then, \( n - 1 \leq k \). By the strong induction hypothesis, \( n - 1 \) can be written as distinct powers of 2. \( n \) is the sum of these same distinct powers of 2 plus \( 2^0 \), which is not among them, since \( n - 1 \) is even.

Case \( n \) is even: Then \( n = 2m \), for \( m \leq k \). Thus, by the strong induction hypothesis, \( m \) can be written as a sum of distinct powers of 2. For each power of 2 in this sum, increment the exponent by 1. The resulting sum of distinct powers of 2 equals \( n = 2m \).
8. (8 points) Let $S$ be the set of bit strings defined recursively by:

(a) $\lambda \in S$, where $\lambda$ is the empty string
(b) $0x \in S, x1 \in S$, if $x \in S$.

Give an explicit description of the elements of $S$.

$S = \{0^i1^j \mid i \geq 0 \text{ and } j \geq 0\}$.